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椭圓抛物面双曲扁壳在均匀外压作用下的非綫性彈性稳定性

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提要： 本文研究了椭圓抛物面双曲扁壳在均匀外压作用下的非綫性弹性稳定問題。文中指出不等曲率双曲扁壳的局部失稳周界可視为一椭圓（等曲率时为一个圓），其半軸长之比与壳面相应的曲率开平方成反比；文中用广义伽辽金变分方程求得了与能量法一致的結果，提出了临界荷載的实用公式，并得到了模型試驗的証实。

前　　言

稳定計算是壳体結构設計的重要依据。近二十余年来，各国力学工作者在用非线性理論研究薄壳稳定性方面作出了不少有价値的貢献[1~7]，特别是在研究軸对称球壳的局部失稳方面提出了許多有益的成果。

椭圓抛物面双曲扁壳在均匀外压作用下的非线性稳定問題，包括总体失稳与局部失稳两类。前者是壳体的非线性强度問題，失稳时出現翻面現象；后者表示壳体失稳发生在薄膜区域，失稳时只局部发生凹陷。

M.A.Колтунов 用伽辽金变分法求得了四边簡支双曲扁壳总体失稳的上下临界荷載[9]。德国 H.Schmidt[10]作了八个鋁制双曲扁壳的模型稳定試驗，試驗結果良好，得出了壳体局部失稳临界荷載的实驗公式，但他对此未作理論上的分析。北京大学数学力学系学生的一篇毕业論文**，給出了双曲扁壳局部失稳的临界荷載公式，由于为了便于运算将失稳曲面視为矩形，所得結果尚不够滿意。

一般建筑工程中常用尺度的双曲扁壳，参数 $\alpha_1=\dfrac{k_x a^2}{h}$ 和 $\alpha_2=\dfrac{k_y b^2}{h}$ 恒大于30。M.A.Колтунов給出 $\alpha_1=\alpha_2=30$ 时，壳面荷載参数 $q^*=\dfrac{qa^2b^2}{Eh^4}$ 与壳頂挠度参数 $\xi=\dfrac{w_c}{h}$ 的关系如图1中的曲线Ⅰ。本文得出 q^*—ξ 曲线如图1中的曲线Ⅱ，点 C 为局部失稳临界荷載。壳体受荷后，壳頂挠度由 O 沿曲线Ⅰ上升到与 C 等高的点 C' 时，壳体局部失稳，发生突然凹陷而跳跃至 C，故壳体总体失稳的上临荷載 U 不会出現；而且 α_i 愈大，U 愈高，U 与 C 相差愈大。因此，局部失稳是設計中的控制因素。

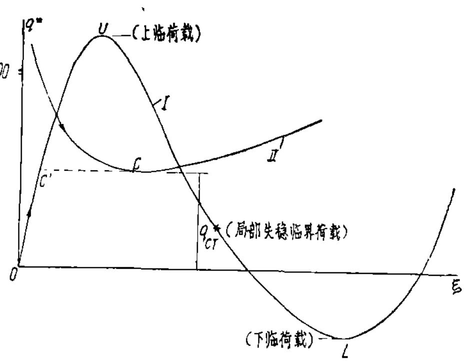

图 1

本文用非线性理論研究了椭圓抛物面双曲扁壳在均匀外压作用下的局部失稳問題。文中用变位法将壳面失稳区之位移 u，v，w 作未知函数，用广义伽辽金变分方程求得与能量法一致的結果，导出了壳体临界荷載的实用公式。文末将模型試驗数据与相应的理

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** 該論文为北京大学数学力学系1963級学生唐軍的毕业論文，論文題目是"均匀外压作用下，不等曲率椭圓抛物型双曲扁壳的弹性稳定性問題"。

論解答作了比較和討論。

一、扁壳非綫性稳定的基本微分方程

扁壳任一点的应力应变关系为（图1.1）

$$\left.\begin{aligned}\sigma_{xz}&=\frac{E}{1-\nu^2}(\varepsilon_{xz}+\nu\varepsilon_{yz}),\\ \sigma_{yz}&=\frac{E}{1-\nu^2}(\varepsilon_{yz}+\nu\varepsilon_{xz}),\\ \tau_z&=\frac{E}{2(1+\nu)}\gamma_z.\end{aligned}\right\}\qquad(1.1)$$

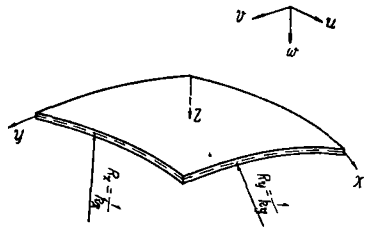


图 1.1

考虑非线性項时，应变与位移的关系式为

$$\left.\begin{aligned}\varepsilon_{xz}&=\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-z\frac{\partial^2 w}{\partial x^2},\\ \varepsilon_{yz}&=\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2-z\frac{\partial^2 w}{\partial y^2},\\ \gamma_z&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2k_{xy}w+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-2z\frac{\partial^2 w}{\partial x\partial y}.\end{aligned}\right\}\qquad(1.2)$$

式中k_x，k_y，k_{xy}分别为壳体中面的曲率及扭曲率。

对应于式（1.2），壳体截面內力公式为

$$\left.\begin{aligned}T_x&=\frac{Eh}{1-\nu^2}\left\{\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\nu\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\right\},\\ T_y&=\frac{Eh}{1-\nu^2}\left\{\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\nu\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\},\\ S&=\frac{Eh}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2k_{xy}w+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right);\end{aligned}\right\}\qquad(1.3)$$

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\ M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),\ M_{xy}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}.\qquad(1.4)$$

式中

$$D=\frac{Eh^3}{12(1-\nu^2)},\qquad(1.5)$$

壳面仅有法向荷載 q 作用时，扁壳的平衡条件归結为

$$\left.\begin{aligned}&\frac{\partial T_x}{\partial x}+\frac{\partial S}{\partial y}=0,\qquad \frac{\partial S}{\partial x}+\frac{\partial T_y}{\partial y}=0,\\ &\left(k_x+\frac{\partial^2 w}{\partial x^2}\right)T_x+\left(k_y+\frac{\partial^2 w}{\partial y^2}\right)T_y+2\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)S+\frac{\partial^2 M_x}{\partial x^2}+\\ &\qquad+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=0.\end{aligned}\right\}\qquad(1.6)$$

将式（1.3）与（1.4）代入式（1.6），整理后，得扁壳的平衡方程

$$\begin{aligned}&\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}-w\frac{\partial}{\partial x}(k_x+\nu k_y)-(k_x+\nu k_y)\frac{\partial w}{\partial x}-(1-\nu)k_{xy}\frac{\partial w}{\partial y}-\\ &-(1-\nu)w\frac{\partial k_{xy}}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}=0;\end{aligned}\qquad(1.7)$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}-w\frac{\partial}{\partial y}(k_y+\nu k_x)-(k_y+\nu k_x)\frac{\partial w}{\partial y}-(1-\nu)k_{xy}\frac{\partial w}{\partial x}-$$

$$-(1-\nu)w\frac{\partial k_{xy}}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}=0; \tag{1.8}$$

$$\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial u}{\partial x}+(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}+k_{xy}\right)\frac{\partial u}{\partial y}+\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\right.$$

$$\left.+\nu\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial v}{\partial y}+(1-\nu)\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)\frac{\partial v}{\partial x}-\left[k_x^2+k_y^2+2(1-\nu)k_{xy}\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)+\right.$$

$$\left.+(k_x+\nu k_y)\frac{\partial^2 w}{\partial x^2}+(k_y+\nu k_x)\frac{\partial^2 w}{\partial y^2}+2\nu k_x k_y\right]w-\frac{h^2}{12}\Delta^2 w+\frac{1}{2}\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\right.$$

$$\left.+\nu\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial y}\right)^2+$$

$$+(1-\nu)\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+q\frac{1-\nu^2}{Eh}=0. \tag{1.9}$$

式中 $$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\qquad \Delta^2=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}. \tag{1.10}$$

扁壳失稳前，中面任一点的位移以u_0，v_0，w_0表示，它們满足式（1.7）～（1.9）。壳体失稳时壳面发生凹陷，失稳区任一点产生新的位移u，v，w，任一点的总位移$u+u_0$，$v+v_0$，$w+w_0$仍满足方程（1.7）～（1.9），得

$$\frac{\partial^2(u+u_0)}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2(u+u_0)}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2(v+v_0)}{\partial x\partial y}-(w+w_0)\frac{\partial}{\partial x}(k_x+\nu k_y)-$$

$$-(k_x+\nu k_y)\frac{\partial(w+w_0)}{\partial x}-(1-\nu)k_{xy}\frac{\partial(w+w_0)}{\partial y}-(1-\nu)(w+w_0)\frac{\partial k_{xy}}{\partial y}+$$

$$+\frac{\partial(w+w_0)}{\partial x}\frac{\partial^2(w+w_0)}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial(w+w_0)}{\partial x}\frac{\partial^2(w+w_0)}{\partial y^2}+$$

$$+\frac{1+\nu}{2}\frac{\partial(w+w_0)}{\partial y}\frac{\partial^2(w+w_0)}{\partial x\partial y}=0; \tag{1.11}$$

$$\frac{\partial^2(v+v_0)}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2(v+v_0)}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2(u+u_0)}{\partial x\partial y}-(w+w_0)\frac{\partial}{\partial y}(k_y+\nu k_x)-$$

$$-(k_y+\nu k_x)\frac{\partial(w+w_0)}{\partial y}-(1-\nu)k_{xy}\frac{\partial(w+w_0)}{\partial x}-(1-\nu)(w+w_0)\frac{\partial k_{xy}}{\partial x}+\frac{\partial(w+w_0)}{\partial y}\times$$

$$\times\frac{\partial^2(w+w_0)}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial(w+w_0)}{\partial y}\frac{\partial^2(w+w_0)}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial(w+w_0)}{\partial x}\frac{\partial^2(w+w_0)}{\partial x\partial y}=0; \tag{1.12}$$

$$\left[k_x+\nu k_y+\frac{\partial^2(w+w_0)}{\partial x^2}+\nu\frac{\partial^2(w+w_0)}{\partial y^2}\right]\frac{\partial(u+u_0)}{\partial x}+(1-\nu)\left[k_{xy}+\frac{\partial^2(w+w_0)}{\partial x\partial y}\right]\frac{\partial(u+u_0)}{\partial y}+$$

$$+\left[k_y+\nu k_x+\frac{\partial^2(w+w_0)}{\partial y^2}+\nu\frac{\partial^2(w+w_0)}{\partial x^2}\right]\frac{\partial(v+v_0)}{\partial y}+(1-\nu)\left[k_{xy}+\frac{\partial^2(w+w_0)}{\partial x\partial y}\right]\times$$

$$\times\frac{\partial(v+v_0)}{\partial x}-\left\{k_x^2+k_y^2+2(1-\nu)k_{xy}\left[k_{xy}+\frac{\partial^2(w+w_0)}{\partial x\partial y}\right]+(k_x+\nu k_y)\frac{\partial^2(w+w_0)}{\partial x^2}+\right.$$

$$\left.+(k_y+\nu k_x)\frac{\partial^2(w+w_0)}{\partial y^2}+2\nu k_x k_y\right\}(w+w_0)-\frac{h^2}{12}\Delta^2(w+w_0)+\frac{1}{2}\left[k_x+\nu k_y+\frac{\partial^2(w+w_0)}{\partial x^2}+\right.$$

$$\left.+\nu\frac{\partial^2(w+w_0)}{\partial y^2}\right]\left[\frac{\partial(w+w_0)}{\partial x}\right]^2+\frac{1}{2}\left[k_y+\nu k_x+\frac{\partial^2(w+w_0)}{\partial y^2}+\nu\frac{\partial^2(w+w_0)}{\partial x^2}\right]\times$$

$$\times\left[\frac{\partial(w+w_0)}{\partial y}\right]^2+(1-\nu)\left[k_{xy}+\frac{\partial^2(w+w_0)}{\partial x\partial y}\right]\frac{\partial(w+w_0)}{\partial y}\frac{\partial(w+w_0)}{\partial x}+q\frac{1-\nu^2}{Eh}=0. \tag{1.13}$$

壳体丧失稳定性（局部失稳），一般发生在薄膜区域，在此区域内的薄膜法向位移 w_0 可視为近似地滿足

$$\frac{\partial w_0}{\partial x}\approx\frac{\partial w_0}{\partial y}\approx 0,\qquad \frac{\partial^2 w_0}{\partial x^2}\approx\frac{\partial^2 w_0}{\partial y^2}\approx\frac{\partial^2 w_0}{\partial x\partial y}\approx 0. \tag{1.14}$$

注意到失稳前壳面任一点位移u_0、v_0、w_0滿足方程（1.7）～（1.9），并与薄膜内力之間存在

$$\left.\begin{aligned}
T_x^0&=\frac{Eh}{1-\nu^2}\left[\left(\frac{\partial u_0}{\partial x}-k_x w_0\right)+\nu\left(\frac{\partial v_0}{\partial y}-k_y w_0\right)\right],\\
T_y^0&=\frac{Eh}{1-\nu^2}\left[\left(\frac{\partial v_0}{\partial y}-k_y w_0\right)+\nu\left(\frac{\partial u_0}{\partial x}-k_x w_0\right)\right],\\
S^0&=\frac{Eh}{2(1+\nu)}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}-2k_{xy}w_0\right),
\end{aligned}\right\} \tag{1.15}$$

因此，从式（1.11）～（1.13）中減去失稳前u_0，v_0，w_0所滿足的方程（1.7）～（1.9），整理后得

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}-w\frac{\partial}{\partial x}(k_x+\nu k_y)-(k_x+\nu k_y)\frac{\partial w}{\partial x}-(1-\nu)k_{xy}\frac{\partial w}{\partial y}-\\
&\quad-(1-\nu)w\frac{\partial k_{xy}}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}=0;
\end{aligned} \tag{1.16}$$

$$\begin{aligned}
&\frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}-w\frac{\partial}{\partial y}(k_y+\nu k_x)-(k_y+\nu k_x)\frac{\partial w}{\partial y}-(1-\nu)k_{xy}\frac{\partial w}{\partial x}-\\
&\quad-(1-\nu)w\frac{\partial k_{xy}}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}=0;
\end{aligned} \tag{1.17}$$

$$\begin{aligned}
&\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial u}{\partial x}+(1-\nu)\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)\frac{\partial u}{\partial y}+\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\right.\\
&\quad\left.+\nu\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial v}{\partial y}+(1-\nu)\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)\frac{\partial v}{\partial x}-\left[k_x^2+k_y^2+2(1-\nu)k_{xy}\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)+\right.\\
&\quad\left.+(k_x+\nu k_y)\frac{\partial^2 w}{\partial x^2}+(k_y+\nu k_x)\frac{\partial^2 w}{\partial y^2}+2\nu k_x k_y\right]w-\frac{h^2}{12}\Delta^2 w+\frac{1}{2}\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\right.\\
&\quad\left.+\nu\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial y}\right)^2+(1-\nu)\left(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\right)\times\\
&\quad\times\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{1-\nu^2}{Eh}\left[T_x^0\frac{\partial^2 w}{\partial x^2}+T_y^0\frac{\partial^2 w}{\partial y^2}+2S^0\frac{\partial^2 w}{\partial x\partial y}\right]=0.
\end{aligned} \tag{1.18}$$

此即扁壳失稳的非线性基本微分方程。

A.C.Вольмир 进行球壳的稳定实驗时，用快速照相机攝得了失稳过程的照片[5]，它表明壳体在开始失稳的瞬間，失稳区域是較小的，隨后才相应地增大，估計这可能是由于实驗时，加荷不能迅速停止等因素的影响所造成的。

对于k_x，k_y为常数，$k_{xy}=0$ 的橢圓拋物面双曲扁壳，在均匀外压作用下的局部失稳将发生在中央薄膜区域。由于失稳区較小，在失稳前的薄膜内力值可近似取[8]

$$T_x^0\approx-\frac{q}{2k_x},\quad T_y^0\approx-\frac{q}{2k_y},\quad S^0\approx 0. \tag{1.19}$$

式（1.16）～（1.18）分别簡化为

$$
\left.
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}-(k_x+\nu k_y)\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\\
&\quad+\frac{1-\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}=0,\\
&\frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}-(k_y+\nu k_x)\frac{\partial w}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\\
&\quad+\frac{1-\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}=0,\\
&\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial u}{\partial x}+(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\\
&+\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial v}{\partial y}-\Big[k_x^2+k_y^2+(k_x+\nu k_y)\frac{\partial^2 w}{\partial x^2}+\\
&+(k_y+\nu k_x)\frac{\partial^2 w}{\partial y^2}+2\nu k_x k_y\Big]w-\frac{h^2}{12}\Delta^2 w+\frac{1}{2}\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\times\\
&\times\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial y}\right)^2+(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\times\\
&\times\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{1-\nu^2}{2Eh}q\left(\frac{1}{k_x}\frac{\partial^2 w}{\partial x^2}+\frac{1}{k_y}\frac{\partial^2 w}{\partial y^2}\right)=0.
\end{aligned}
\right\}
\qquad (1.20)
$$

对于不等曲率双曲扁壳（$k_x \neq k_y$），壳体失稳曲面周界之水平投影将为一椭圆（图1.2）。将其边界近似視为嵌固[5]，并将座标原点取在椭圆中心，則边界曲线方程为

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (1.21)$$

式中 a 与 b 分别表示失稳曲面椭圆在 x 及 y 方向之半軸长。

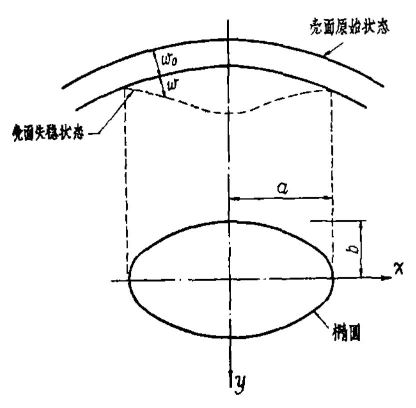


图1.2

对于等曲率双曲扁壳（$k_x=k_y$），$a=b$，式（1.21）退化为圆

$$x^2+y^2=a^2. \qquad (1.22)$$

由于壳体系局部失稳，可不計壳体实际边界对壳体失稳的影响，所需考虑的边界条件为其失稳区的边界条件，当近似取为嵌固时，边界条件为

$$u_s=v_s=w_s=\left(\frac{\partial w}{\partial n}\right)_s=0. \qquad (1.23)$$

解式（1.20），并使未知函数滿足边界条件（1.23），可以求出壳体的临界荷載 q_{cr}，但式（1.20）为非线性偏微分方程組，直接求解頗为困难。本文将采用广义伽辽金变分方程及能量法来导出壳体临界荷載的实用公式。

二、壳体失稳区位移函数的表达式

我們近似地取壳面失稳区法向位移函数

$$w=A\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2, \qquad (2.1)$$

它滿足边界条件

$$w_s=\left(\frac{\partial w}{\partial n}\right)_s=0. \qquad (2.2)$$

将式（2.1）代入式（1.20）前两式的切向力平衡方程求解，并使边界条件 $u_s=v_s=0$ 得到滿足，可得壳面失稳区切向位移函数

$$u=\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(B_1\frac{x}{a}+B_2\frac{x^3}{a^3}+B_3\frac{xy^2}{ab^2}+B_4\frac{x^5}{a^5}+2B_5\frac{x^3y^2}{a^3b^2}+B_6\frac{xy^4}{ab^4}\right), \tag{2.3}$$

$$v=\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(C_1\frac{y}{b}+C_2\frac{y^3}{b^3}+C_3\frac{x^2y}{a^2b}+C_4\frac{y^5}{b^5}+2C_5\frac{x^2y^3}{a^2b^3}+C_6\frac{x^4y}{a^4b}\right). \tag{2.4}$$

将式（2.1）、（2.3）及（2.4）代入式（1.20）前两式，利用比较系数的方法，得到确定系数 B_i，$C_i(i=1,\ 2,\ \cdots\cdots,\ 6)$的代数方程組

$$\left.\begin{aligned}
&-\left(\frac{6}{a^3}+\frac{1-\nu}{ab^2}\right)B_1+\frac{6}{a^3}B_2+\frac{1-\nu}{ab^2}B_3-\frac{1+\nu}{a^2b}C_1+\frac{1+\nu}{a^2b}C_3=\\
&\qquad=-\left[\frac{4A(k_x+\nu k_y)}{a^2}+\frac{16A^2}{a^4}+\frac{8A^2(1-\nu)}{a^2b^2}\right],\\
&-\frac{6}{a^3b^2}B_2-6\left(\frac{1}{a^3b^2}+\frac{1-\nu}{ab^4}\right)B_3+\frac{12}{a^3b^2}B_5+\frac{6(1-\nu)}{ab^4}B_6-\frac{3(1+\nu)}{a^2b^3}C_2-\frac{3(1+\nu)}{a^2b^3}C_3+\\
&\qquad+\frac{6(1+\nu)}{a^2b^3}C_5=\frac{4A(k_x+\nu k_y)}{a^2b^2}+\frac{32A^2}{a^4b^2}+\frac{32A^2(1-\nu)}{a^2b^4}+\frac{16A^2(1+\nu)}{a^2b^4},\\
&-\left(\frac{20}{a^5}+\frac{1-\nu}{a^3b^2}\right)B_2-\frac{1-\nu}{a^3b^2}B_3+\frac{20}{a^5}B_4+\frac{2(1-\nu)}{a^3b^2}B_5-\frac{2(1+\nu)}{a^4b}C_3+\frac{2(1+\nu)}{a^4b}C_6=\\
&\qquad=\frac{4A(k_x+\nu k_y)}{a^4}+\frac{64A^2}{a^6}+\frac{16A^2(1-\nu)}{a^4b^2},\\
&\frac{12}{a^3b^4}B_5+\left(\frac{6}{a^3b^4}+\frac{15(1-\nu)}{ab^6}\right)B_6+\frac{5(1+\nu)}{a^2b^5}C_4+\frac{10(1+\nu)}{a^2b^5}C_5=\\
&\qquad=\frac{16A^2}{a^4b^4}+\frac{24A^2(1-\nu)}{a^2b^6}+\frac{16A^2(1+\nu)}{a^2b^6},\\
&\frac{20}{a^5b^2}B_4+\left(\frac{40}{a^5b^2}+\frac{12(1-\nu)}{a^3b^4}\right)B_5+\frac{6(1-\nu)}{a^3b^4}B_6+\frac{12(1+\nu)}{a^4b^3}C_5+\frac{6(1+\nu)}{a^4b^3}C_6=\\
&\qquad=\frac{64A^2}{a^6b^2}+\frac{32A^2(1-\nu)}{a^4b^4}+\frac{16A^2(1+\nu)}{a^4b^4},\\
&\left(\frac{42}{a^7}-\frac{1-\nu}{a^5b^2}\right)B_4+\frac{2(1-\nu)}{a^5b^2}B_5+\frac{3(1+\nu)}{a^6b}C_6=\frac{48A^2}{a^8}+\frac{8A^2(1-\nu)}{a^6b^2},\\
&-\frac{1+\nu}{ab^2}B_1+\frac{1+\nu}{ab^2}B_3-\left(\frac{6}{b^3}+\frac{1-\nu}{a^2b}\right)C_1+\frac{6}{b^3}C_2+\frac{1-\nu}{a^2b}C_3=\\
&\qquad=-\left[\frac{4A(k_y+\nu k_x)}{b^2}+\frac{16A^2}{b^4}+\frac{8A^2(1-\nu)}{a^2b^2}\right],\\
&-\frac{3(1+\nu)}{a^3b^2}B_2-\frac{3(1+\nu)}{a^3b^2}B_3+\frac{6(1+\nu)}{a^3b^2}B_5-\frac{6}{a^2b^3}C_2-6\left(\frac{1}{a^2b^3}+\frac{1-\nu}{a^4b}\right)C_3+\\
&\qquad+\frac{12}{a^2b^3}C_5+\frac{6(1-\nu)}{a^4b}C_6=\frac{4A(k_y+\nu k_x)}{a^2b^2}+\frac{32A^2}{a^2b^4}+\frac{32A^2(1-\nu)}{a^4b^2}+\frac{16A^2(1+\nu)}{a^4b^2},\\
&-\frac{2(1+\nu)}{ab^4}B_3+\frac{2(1+\nu)}{ab^4}B_6-\left(\frac{20}{b^5}+\frac{1-\nu}{a^2b^3}\right)C_2-\frac{1-\nu}{a^2b^3}C_3+\frac{20}{b^5}C_4+\frac{2(1-\nu)}{a^2b^3}C_5=\\
&\qquad=\frac{4A(k_y+\nu k_x)}{b^4}+\frac{64A^2}{b^6}+\frac{16A^2(1-\nu)}{a^2b^4},
\end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned}
&\frac{5(1+\nu)}{a^5b^2}B_4+\frac{10(1+\nu)}{a^5b^2}B_5+\frac{12}{a^4b^3}C_5+\left(\frac{6}{a^4b^3}+\frac{15(1-\nu)}{a^6b}\right)C_6=\\
&\qquad=\frac{16A^2}{a^4b^4}+\frac{24A^2(1-\nu)}{a^6b^2}+\frac{16A^2(1+\nu)}{a^6b^2},\\
&\frac{12(1+\nu)}{a^3b^4}B_5+\frac{6(1+\nu)}{a^3b^4}B_6+\frac{20}{a^2b^5}C_4+\left(\frac{40}{a^2b^5}+\frac{12(1-\nu)}{a^4b^3}\right)C_5+\frac{6(1-\nu)}{a^4b^3}C_6=\\
&\qquad=\frac{64A^2}{a^2b^6}+\frac{32A^2(1-\nu)}{a^4b^4}+\frac{16A^2(1+\nu)}{a^4b^4},\\
&\frac{3(1+\nu)}{ab^6}B_6+\left(\frac{42}{b^7}+\frac{1-\nu}{a^2b^5}\right)C_4+\frac{2(1-\nu)}{a^2b^5}C_5=\frac{48A^2}{b^8}+\frac{8A^2(1-\nu)}{a^2b^6}.
\end{aligned}\right\}$$

令

$$\left.\begin{aligned}
&A=h\xi,\qquad B_i=\widetilde{B}_ih,\qquad C_i=\widetilde{C}_ih,\qquad (i=1,2,\cdots\cdots,6);\\
&\quad a=K_1\eta,\qquad b=K_2\eta.
\end{aligned}\right\}\tag{2.6}$$

式中 ξ 为无量綱挠度参数；η 为壳面失稳区无量綱軸长参数；K_1 与 K_2 为与曲率及壳厚有关的系数，其量綱为长度。

将式（2.6）代入（2.5），整理后，得到无量綱代数方程組

$$\left.\begin{aligned}
&-\left(6+\frac{K_1^2(1-\nu)}{K_2^2}\right)\widetilde{B}_1+6\widetilde{B}_2+\frac{K_1^2}{K_2^2}(1-\nu)\widetilde{B}_3-\frac{K_1}{K_2}(1+\nu)\widetilde{C}_1+\frac{K_1}{K_2}(1+\nu)\widetilde{C}_3=\\
&\qquad=-\left[4K_1(k_x+\nu k_y)\xi\eta+\frac{\xi^2}{\eta}\left(\frac{16h}{K_1}+\frac{8K_1h(1-\nu)}{K_2^2}\right)\right],\\
&-\left(20+\frac{K_1^2(1-\nu)}{K_2^2}\right)\widetilde{B}_2-\frac{K_1^2}{K_2^2}(1-\nu)\widetilde{B}_3+20\widetilde{B}_4+\frac{2K_1^2}{K_2^2}(1-\nu)\widetilde{B}_5-2(1+\nu)\frac{K_1}{K_2}\widetilde{C}_3+\\
&\qquad+2(1+\nu)\frac{K_1}{K_2}\widetilde{C}_6=4K_1(k_x+\nu k_y)\xi\eta+\frac{\xi^2}{\eta}\left(\frac{64h}{K_1}-\frac{16K_1h(1-\nu)}{K_1^2}\right),\\
&\left(42+\frac{K_1^2(1-\nu)}{K_2^2}\right)\widetilde{B}_4+2(1-\nu)\frac{K_1^2}{K_2^2}\widetilde{B}_5+3(1+\nu)\frac{K_1}{K_2}\widetilde{C}_6=\frac{\xi^2}{\eta}\left(\frac{48h}{K_1}+\frac{8K_1h(1-\nu)}{K_2^2}\right),\\
&-6\widetilde{B}_2-\left(6+\frac{6K_1^2(1-\nu)}{K_2^2}\right)\widetilde{B}_3+12\widetilde{B}_5+6(1-\nu)\frac{K_1^2}{K_2^2}\widetilde{B}_6-3(1+\nu)\frac{K_1}{K_2}\widetilde{C}_2-3(1+\nu)\frac{K_1}{K_2}\widetilde{C}_3\\
&\qquad+6(1+\nu)\frac{K_1}{K_2}\widetilde{C}_5=4K_1(k_x+\nu k_y)\xi\eta+\frac{\xi^2}{\eta}\left(\frac{32h}{K_1}+\frac{48K_1h}{K_2^2}-\frac{16\nu K_1h}{K_2^2}\right),\\
&20\widetilde{B}_4+\left(40+\frac{12K_1^2(1-\nu)}{K_2^2}\right)\widetilde{B}_5+6(1-\nu)\frac{K_1^2}{K_2^2}\widetilde{B}_6+12(1+\nu)\frac{K_1}{K_2}\widetilde{C}_5+6(1+\nu)\frac{K_1}{K_2}\widetilde{C}_6=\\
&\qquad=\frac{\xi^2}{\eta}\left(\frac{64h}{K_1}+\frac{48K_1h}{K_2^2}-\frac{16\nu K_1h}{K_2^2}\right),\\
&12\widetilde{B}_5+\left(6+\frac{15K_1^2(1-\nu)}{K_2^2}\right)\widetilde{B}_6+5(1+\nu)\frac{K_1}{K_2}\widetilde{C}_4+10(1+\nu)\frac{K_1}{K_2}\widetilde{C}_5=\\
&\qquad=\frac{\xi^2}{\eta}\left(\frac{16h}{K_1}+\frac{40K_1h}{K_2^2}-\frac{8\nu K_1h}{K_2^2}\right),\\
&-(1+\nu)\frac{K_2}{K_1}\widetilde{B}_1+(1+\nu)\frac{K_2}{K_1}\widetilde{B}_3-\left(6+\frac{K_2^2(1-\nu)}{K_1^2}\right)\widetilde{C}_1+6\widetilde{C}_2+(1-\nu)\frac{K_2^2}{K_1^2}\widetilde{C}_3=\\
&\qquad=-\left[4K_2(k_y+\nu k_x)\xi\eta+\frac{\xi^2}{\eta}\left(\frac{16h}{K_2}+\frac{8K_2h(1-\nu)}{K_1^2}\right)\right],\\
&-2(1+\nu)\frac{K_2}{K_1}\widetilde{B}_3+2(1+\nu)\frac{K_2}{K_1}\widetilde{B}_6-\left(20+\frac{K_2^2(1-\nu)}{K_1^2}\right)\widetilde{C}_2-(1-\nu)\frac{K_2^2}{K_1^2}\widetilde{C}_3+20\widetilde{C}_4+
\end{aligned}\right\}\tag{2.7}$$

$$
\begin{aligned}
&+2(1-\nu)\frac{K_2^2}{K_1^2}\widetilde{C}_5=4K_2(k_y+\nu k_x)\xi\eta+\frac{\xi^2}{\eta}\left(\frac{64h}{K_2}+\frac{16K_2h(1-\nu)}{K_1^2}\right),\\
&3(1+\nu)\frac{K_2}{K_1}\widetilde{B}_6+\left(42+\frac{K_2^2(1-\nu)}{K_1^2}\right)\widetilde{C}_4+2(1-\nu)\frac{K_2^2}{K_1^2}\widetilde{C}_5=\frac{\xi^2}{\eta}\left(\frac{48h}{K_2}+\frac{8K_2h(1-\nu)}{K_1^2}\right),\\
&-3(1+\nu)\frac{K_2}{K_1}\widetilde{B}_2-3(1+\nu)\frac{K_2}{K_1}\widetilde{B}_3+6(1+\nu)\frac{K_2}{K_1}\widetilde{B}_5-6\widetilde{C}_2-\left(6+\frac{6K_2^2(1-\nu)}{K_1^2}\right)\widetilde{C}_3+\\
&+12\widetilde{C}_5+6(1-\nu)\frac{K_2^2}{K_1^2}\widetilde{C}_6=4K_2(k_y+\nu k_x)\xi\eta+\frac{\xi^2}{\eta}\left(\frac{32h}{K_2}+\frac{48K_2h}{K_1^2}-\frac{16\nu K_2h}{K_1^2}\right),\\
&12(1+\nu)\frac{K_2}{K_1}\widetilde{B}_5+6(1+\nu)\frac{K_2}{K_1}\widetilde{B}_6+20\widetilde{C}_4+\left(40+\frac{12K_2^2(1-\nu)}{K_1^2}\right)\widetilde{C}_5+6(1-\nu)\frac{K_2^2}{K_1^2}\widetilde{C}_6=\\
&\qquad=\frac{\xi^2}{\eta}\left(\frac{64h}{K_2}+\frac{48K_2h}{K_1^2}-\frac{16\nu K_2h}{K_1^2}\right),\\
&5(1+\nu)\frac{K_2}{K_1}\widetilde{B}_4+10(1+\nu)\frac{K_2}{K_1}\widetilde{B}_5+12\widetilde{C}_5+\left(6+\frac{15K_2^2(1-\nu)}{K_1^2}\right)\widetilde{C}_6\\
&\qquad=\frac{\xi^2}{\eta}\left(\frac{16h}{K_2}+\frac{40K_2h}{K_1^2}-\frac{8\nu K_2h}{K_1^2}\right).
\end{aligned}
$$

三、用广义伽辽金变分法計算橢圆抛物面双曲扁壳的稳定性

我們給出扁壳非线性稳定的势能变分原理，定义泛函

$$
\begin{aligned}
I=&\iint_F\Big\{\frac{D}{2}\Big[\Big(\frac{\partial^2w}{\partial x^2}\Big)^2+\Big(\frac{\partial^2w}{\partial y^2}\Big)^2+2\nu\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}+2(1-\nu)\Big(\frac{\partial^2w}{\partial x\partial y}\Big)^2\Big]+\\
&+\frac{Eh}{2(1-\nu^2)}\Big[\Big(\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big)^2+\Big(\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big)^2+\\
&+2\nu\Big(\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big)\Big(\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big)+\frac{1-\nu}{2}\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2k_{xy}w+\\
&+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big)^2\Big]+T_x^0\Big[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big]+T_y^0\Big[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big]+\\
&+S^0\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2k_{xy}w+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big)-qw\Big\}dxdy-\\
&-\int_{C_{T_n}}\overline{T}_nu_nds-\int_{C_{T_{ns}}}\overline{T}_{ns}u_sds-\int_{C_{R_n}}\Big(\overline{R}_n+\overline{T}_n\frac{\partial w}{\partial n}+\overline{T}_{ns}\frac{\partial w}{\partial s}\Big)wds+\int_{C_m}\overline{M}_n\frac{\partial w}{\partial n}ds.
\end{aligned}
\qquad (3.1)
$$

式中代号 n,s 表示壳体失稳区域边界之法向及切向座标。C_{T_n} 、$C_{T_{ns}}$、C_{R_n} 、C_m为应力边界；$\overline{T}_n$、$\overline{T}_{ns}$、$\overline{R}_n$、$\overline{M}_n$ 为相应边界上給定的应力值，可表示为

$$
\left.
\begin{aligned}
&\text{在 } C_{T_n} \text{ 上,} && T_n=T_xl^2+2Slm+T_ym^2=\overline{T}_n,\\
&\text{在 } C_{T_{ns}} \text{ 上,} && T_{ns}=(T_y-T_x)lm+S(l^2-m^2)=\overline{T}_{ns},\\
&\text{在 } C_{R_n} \text{ 上,} && R_n=Q_n+\frac{\partial}{\partial s}M_{ns}=\overline{R}_n,\\
&\text{在 } C_m \text{ 上,} && M_n=M_xl^2+2M_{xy}lm+M_ym^2=\overline{M}_n.
\end{aligned}
\right\}
\qquad (3,2)
$$

式中 $$l=\cos(n,x),\qquad m=\cos(n,y). \qquad (3.3)$$

在扁壳稳定間題中，式（3.1）之积分限作为变量处理，失稳区域不变的情况可視为其特例。对式（3.1）作变分，并令

$$\delta I=0, \qquad (3.4)$$

得

$$\frac{Eh}{1-\nu^2}\iint_F\Big\{-\frac{\partial u}{\partial x}\Big(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\Big)-\frac{\partial v}{\partial y}\Big(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\Big)+$$

$$+w\Big[k_x^2+k_y^2+2k_xk_y\nu+2k_{xy}^2(1-\nu)+k_x\frac{\partial^2 w}{\partial x^2}+k_y\frac{\partial^2 w}{\partial y^2}+\nu\Big(k_x\frac{\partial^2 w}{\partial y^2}+k_y\frac{\partial^2 w}{\partial x^2}\Big)+$$

$$+2k_{xy}(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\Big]-\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\Big)-\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big(k_y+\nu k_x+$$

$$+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\Big)+\frac{h^2}{12}\Delta^2 w-(1-\nu)\frac{\partial u}{\partial y}\Big(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\Big)-\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}(1-\nu)\Big(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\Big)-$$

$$-(1-\nu)\frac{\partial v}{\partial x}\Big(k_{xy}+\frac{\partial^2 w}{\partial x\partial y}\Big)-\frac{1-\nu^2}{Eh}\Big[T_x^0\frac{\partial^2 w}{\partial x^2}+T_y^0\frac{\partial^2 w}{\partial y^2}+2S^0\frac{\partial^2 w}{\partial x\partial y}\Big]\Big\}\delta w\,dxdy-$$

$$-\frac{Eh}{1-\nu^2}\iint_F\Big[\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}-w\frac{\partial}{\partial x}(k_x+\nu k_y)-(k_x+\nu k_y)\frac{\partial w}{\partial x}-$$

$$-(1-\nu)k_{xy}\frac{\partial w}{\partial y}-(1-\nu)w\frac{\partial k_{xy}}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big]\times$$

$$\times\Big(\delta u+\frac{\partial w}{\partial x}\delta w\Big)dxdy-\frac{Eh}{1-\nu^2}\iint_F\Big[\frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}-w\frac{\partial}{\partial y}(k_y+\nu k_x)-$$

$$-(k_y+\nu k_x)\frac{\partial w}{\partial y}-(1-\nu)k_{xy}\frac{\partial w}{\partial x}-(1-\nu)w\frac{\partial k_{xy}}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}+$$

$$+\frac{1+\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\Big]\Big(\delta v+\frac{\partial w}{\partial y}\delta w\Big)dxdy+\int_{C_w}\Big(R_n+T_n\frac{\partial w}{\partial n}+T_{ns}\frac{\partial w}{\partial s}+T_n^0\frac{\partial w}{\partial n}+$$

$$+T_{ns}^0\frac{\partial w}{\partial s}\Big)\delta w\,ds-\int_{C_\alpha}M_n\delta\Big(\frac{\partial w}{\partial n}\Big)ds+\int_{C_{u_n}}(T_n+T_n^0)\,\delta u_n ds+\int_{C_{u_s}}(T_{ns}+T_{ns}^0)\,\delta u_s ds+$$

$$+\int_{C_{T_n}}(T_n+T_n^0-\overline{T}_n)\delta u_n ds+\int_{C_{T_{ns}}}(T_{ns}+T_{ns}^0-\overline{T}_{ns})\delta u_s ds+\int_{C_{R_n}}\Big[R_n+(T_n+T_n^0)\frac{\partial w}{\partial n}+$$

$$+(T_{ns}+T_{ns}^0)\frac{\partial w}{\partial s}-\overline{R}_n-\overline{T}_n\frac{\partial w}{\partial n}-\overline{T}_{ns}\frac{\partial w}{\partial s}\Big]\delta w\,ds-\int_{C_m}(M_n-\overline{M}_n)\frac{\partial\delta w}{\partial n}ds-$$

$$-\iint_F(T_x^0k_x+T_y^0k_y+2k_{xy}S^0+q)\,\delta w\,dxdy+\psi=0. \qquad (3.5)$$

式中C_w、C_α、C_{u_n}、C_{u_s}表示位移边界，ψ为变分时将积分限作变限处理出现的項，ψ的形式可参看(3.10b)式。

我們研究椭圆抛物面双曲扁壳的局部失稳，壳体失稳曲面周界的水平投影为一椭圆($a=K_1\eta, b=K_2\eta$)，失稳区域之边界为位移边界，$k_{xy}=0$，k_x与k_y均为常量，失稳前壳体之薄膜内力如式（1.19）所示。于是，式（3.5）簡化为

$$\frac{Eh}{1-\nu^2}\iint_F\Big\{-\frac{\partial u}{\partial x}\Big(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\Big)-\frac{\partial v}{\partial y}\Big(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\Big)+$$

$$+w\Big[k_x^2+k_y^2+2\nu k_xk_y+k_x\frac{\partial^2 w}{\partial x^2}+k_y\frac{\partial^2 w}{\partial y^2}+\nu\Big(k_x\frac{\partial^2 w}{\partial y^2}+k_y\frac{\partial^2 w}{\partial x^2}\Big)\Big]-\frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\times$$

$$\times\Big(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\Big)-\frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Big(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\Big)+\frac{h^2}{12}\Delta^2 w-$$

$$-(1-\nu)\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big)\frac{\partial^2 w}{\partial x\partial y}-\frac{1-\nu^2}{Eh}\Big[T_x^0\frac{\partial^2 w}{\partial x^2}+T_y^0\frac{\partial^2 w}{\partial y^2}\Big]\Big\}\delta w\,dxdy-$$

$$-\frac{Eh}{1-\nu^2}\iint_F\Big[\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}-(k_x+\nu k_y)\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}+$$

$$+\frac{1+\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big]\Big[\delta u+\frac{\partial w}{\partial x}\delta w\Big]dxdy-\frac{Eh}{1-\nu^2}\iint_F\Big[\frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}-$$

$$-(k_y+\nu k_x)\frac{\partial w}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\Big]\Big(\delta v+\frac{\partial w}{\partial y}\delta w\Big)dxdy+$$

$$+\int_C\Big(R_n+T_n\frac{\partial w}{\partial n}+T_{ns}\frac{\partial w}{\partial s}+T_n^0\frac{\partial w}{\partial n}+T_{ns}^0\frac{\partial w}{\partial s}\Big)\delta w ds-\int_C M_n\delta\Big(\frac{\partial w}{\partial n}\Big)ds+$$

$$+\int_C(T_n+T_n^0)\delta u_n ds+\int_C(T_{ns}+T_{ns}^0)\delta u_s ds+\overline{\psi}=0. \qquad (3.6)$$

式中 $\overline{\psi}$为积分限作变限处理而出现的项。

若边界处之位移变分及$\overline{\psi}$值均为零，式（3.6）即化为常用的伽辽金方程。为資区别，作者将式(3.6)称为广义伽辽金变分方程。

将壳体失稳区域之位移函数表为滿足位移边界条件（1.23）的公式（2.1）、（2.3）及（2.4），壳体失稳区域之积分限为

$$x\text{ 自 }0\text{ 至 }K_1\eta,\qquad y\text{ 自 }0\text{ 至 }\Big(\sqrt{K_1^2\eta^2-x^2}\Big)\frac{K_2}{K_1},$$

将A、η、B_i 与 C_i 視为相互独立之变量，得

$$\left.\begin{aligned}
\delta w=&\Big(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\Big)^2\delta A+4A\Big(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\Big)\Big(\frac{x^2}{K_1^2\eta^3}+\frac{y^2}{K_2^2\eta^3}\Big)\delta\eta,\\
\delta\Big(\frac{\partial w}{\partial x}\Big)=&-4\Big(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\Big)\frac{x}{K_1^2\eta^2}\delta A+4A\Big(\frac{2x}{K_1^2\eta^3}-\frac{4x^3}{K_1^4\eta^5}-\\
&-\frac{4xy^2}{K_1^2K_2^2\eta^5}\Big)\delta\eta,\\
\delta\Big(\frac{\partial w}{\partial y}\Big)=&-4\Big(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\Big)\frac{y}{K_2^2\eta^2}\delta A+4A\Big(\frac{2y}{K_2^2\eta^3}-\frac{4x^2y}{K_1^2K_2^2\eta^5}-\\
&-\frac{4y^3}{K_2^4\eta^5}\Big)\delta\eta,\\
\delta u=&\Big(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\Big)\Big(\frac{x}{K_1\eta}\delta B_1+\frac{x^3}{K_1^3\eta^3}\delta B_2+\frac{xy^2}{K_1K_2^2\eta^3}\delta B_3+\frac{x^5}{K_1^5\eta^5}\delta B_4+\\
&+2\frac{x^3y^2}{K_1^3K_2^2\eta^5}\delta B_5+\frac{xy^4}{K_1K_2^4\eta^5}\delta B_6\Big)-\Big\{B_1\frac{x}{K_1}\Big[\frac{1}{\eta^2}-\frac{3}{\eta^4}\Big(\frac{x^2}{K_1^2}+\frac{y^2}{K_2^2}\Big)\Big]+\\
&+\Big(B_2\frac{x^3}{K_1^3}+B_3\frac{xy^2}{K_1K_2^2}\Big)\Big[\frac{3}{\eta^4}-\frac{5}{\eta^6}\Big(\frac{x^2}{K_1^2}+\frac{y^2}{K_2^2}\Big)\Big]+\Big(B_4\frac{x^5}{K_1^5}+2B_5\frac{x^3y^2}{K_1^3K_2^2}+\\
&+B_6\frac{xy^4}{K_1K_2^4}\Big)\Big[\frac{5}{\eta^6}-\frac{7}{\eta^8}\Big(\frac{x^2}{K_1^2}+\frac{y^2}{K_2^2}\Big)\Big]\Big\}\delta\eta,\\
\delta v=&\Big(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\Big)\Big(\frac{y}{K_2\eta}\delta C_1+\frac{y^3}{K_2^3\eta^3}\delta C_2+\frac{x^2y}{K_1^2K_2\eta^3}\delta C_3+\frac{y^5}{K_2^5\eta^5}\delta C_4+\\
&+2\frac{x^2y^3}{K_1^2K_2^3\eta^5}\delta C_5+\frac{x^4y}{K_1^4K_2}\delta C_6\Big)-\Big\{C_1\frac{y}{K_2}\Big[\frac{1}{\eta^2}-\frac{3}{\eta^4}\Big(\frac{x^2}{K_1^2}+\frac{y^2}{K_2^2}\Big)\Big]+\\
&+\Big(C_2\frac{y^3}{K_2^3}+C_3\frac{x^2y}{K_1^2K_2}\Big)\Big[\frac{3}{\eta^4}-\frac{5}{\eta^6}\Big(\frac{x^2}{K_1^2}+\frac{y^2}{K_2^2}\Big)\Big]+\Big(C_4\frac{y^5}{K_2^5}+2C_5\frac{x^2y^3}{K_1^2K_2^3}+\\
&+C_6\frac{x^4y}{K_1^4K_2}\Big)\Big[\frac{5}{\eta^6}-\frac{7}{\eta^8}\Big(\frac{x^2}{K_1^2}+\frac{y^2}{K_2^2}\Big)\Big]\Big\}\delta\eta.
\end{aligned}\right\}\quad(3.7)$$

边界处之位移变分为

$$\delta\left(\frac{\partial w}{\partial n}\right)=l\delta\left(\frac{\partial w}{\partial x}\right)+m\delta\left(\frac{\partial w}{\partial y}\right), \tag{3.8}$$

$$\delta u_\eta=l\delta u+m\delta v, \quad \delta u_s=-m\delta u+l\delta v. \tag{3.9}$$

由式（3.8）可见，边界处的位移变分 $\delta w=0$，其他位移变分之 δA 項亦为零，其中 $\delta\eta$ 之項均不等于零。

将式（3.7）～（3.9）代入（3.6），并令式（2.3）与（2.4）中之系数 B_i 及 C_i 滿足方程（2.7），变分式进一步簡化为

$$\begin{aligned}
&\frac{Eh}{1-\nu^2}\int_0^{K_1\eta}\int_0^{\frac{K_2}{K_1}\sqrt{K_1^2\eta^2-x^2}}\left\{-\frac{\partial u}{\partial x}\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-\frac{\partial v}{\partial y}\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\right.\right.\\
&\left.+\nu\frac{\partial^2 w}{\partial x^2}\right)+w\left[k_x^2+k_y^2+2\nu k_x k_y+k_x\frac{\partial^2 w}{\partial x^2}+k_y\frac{\partial^2 w}{\partial y^2}+\nu\left(k_x\frac{\partial^2 w}{\partial y^2}+k_y\frac{\partial^2 w}{\partial x^2}\right)\right]-\\
&-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(k_x+\nu k_y+\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\left(k_y+\nu k_x+\frac{\partial^2 w}{\partial y^2}+\right.\\
&\left.+\nu\frac{\partial^2 w}{\partial x^2}\right)+\frac{h^2}{12}\Delta^2 w-(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)+\frac{1-\nu^2}{Eh}\left[\frac{q}{2k_x}\frac{\partial^2 w}{\partial x^2}+\right.\\
&\left.\left.+\frac{q}{2k_y}\frac{\partial^2 w}{\partial y^2}\right]\right\}\left[\left(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\right)\delta A+\underset{\sim\sim\sim}{4A\left(1-\frac{x^2}{K_1^2\eta^2}-\frac{y^2}{K_2^2\eta^2}\right)\left(\frac{x^2}{K_1^2\eta^3}+\right.}\right.\\
&\left.\left.\underset{\sim\sim\sim}{+\frac{y^2}{K_2^2\eta^3}\right)\delta\eta}\right]dydx-\underset{\sim\sim\sim}{\int_c(l^2M_x+2lmM_{xy}+m^2M_y)\left[l\delta\left(\frac{\partial w}{\partial x}\right)+m\delta\left(\frac{\partial w}{\partial y}\right)\right]ds}-\\
&\underset{\sim\sim\sim}{-\int_c[l^2(T_x+T_x^0)+2lmS+m^2(T_y+T_y^0)](l\delta u+m\delta v)ds}+\underset{\sim\sim\sim}{\int_c[lm(T_y+T_y^0-T_x-T_x^0)+}\\
&\underset{\sim\sim\sim}{+S(l^2-m^2)](l\delta v-m\delta u)ds+\bar{\psi}=0.}
\end{aligned}\tag{3.10a}$$

式中划波紋线项为含 $\delta\eta$ 项，其余为含 δA 项。

将式（2.1）、（2.3）与（2.4）之位移函数代入式（3.1）中面积分号下之所有項，并用 $G(x,\ y,\ \eta,\ B_i,\ C_i,\ A)$ 表示，再棄去反映应力边界之各項后，

$$I=\int_0^{K_1\eta}\int_0^{\frac{K_2}{K_1}\sqrt{K_1^2\eta^2-x^2}}G(x,\ y,\ \eta,\ B_i,\ C_i,\ A)dydx.$$

于是，

$$\begin{aligned}
\bar{\psi}=&\left[\int_0^{K_1\eta}G\left(x,\ \frac{K_2}{K_1}\sqrt{K_1^2\eta^2-x^2},\ \eta,\ B_i,\ C_i,\ A\right)\frac{K_1K_2\eta}{\sqrt{K_1^2\eta^2-x^2}}dx+\right.\\
&\left.+K_1\int_0^{\frac{K_2}{K_1}\sqrt{K_1^2\eta^2-x^2}}G(x,\ y,\ \eta,\ B_i,\ C_i,\ A)dy\Big|_{x=K_1\eta}\right]\delta\eta,
\end{aligned}$$

式中之第二項为零，故

$$\bar{\psi}=\left[\int_0^{K_1\eta}G\left(x,\ \frac{K_2}{K_1}\sqrt{K_1^2\eta^2-x^2},\ \eta,\ B_i,\ C_i,\ A\right)\frac{K_1K_2\eta}{\sqrt{K_1^2\eta^2-x^2}}dx\right]\delta\eta. \tag{3.10b}$$

对式（3.10a）作积分运算，利用式（2.6）将所得結果无量綱化，并注意到 $\delta\xi$ 与 $\delta\eta$ 相互独立及壳面失稳时 $\xi\neq0$，我們得到

$$q^*=\frac{1-\nu^2}{E}q=\frac{24k_xh\cdot k_yh}{k_xh\frac{K_1^2}{h^2}+k_yh\frac{K_2^2}{h^2}}\left\{\frac{1}{6\eta^2}\left(\frac{K_2^2}{2K_1^2}+\frac{K_1^2}{2K_2^2}+\frac{1}{3}\right)+\frac{1}{\eta}\left[\frac{K_2^2}{2K_1h}\left(\frac{\widetilde{B}_2}{30}+\frac{\widetilde{B}_4}{42}+\frac{\widetilde{B}_5}{210}\right)+\right.\right.$$

$$+\frac{K_1^2}{2K_2h}\left(\frac{\widetilde{C}_2}{30}+\frac{\widetilde{C}_4}{42}+\frac{\widetilde{C}_5}{210}\right)+\frac{\nu K_1}{4h}\left(\frac{\widetilde{B}_1}{5}+\frac{\widetilde{B}_2}{20}+\frac{\widetilde{B}_3}{60}+\frac{3\widetilde{B}_4}{168}+\frac{\widetilde{B}_5}{140}+\frac{\widetilde{B}_6}{280}\right)+\frac{\nu K_2}{4h}\times$$

$$\times\left(\frac{\widetilde{C}_1}{5}+\frac{\widetilde{C}_2}{20}+\frac{\widetilde{C}_3}{60}+\frac{3\widetilde{C}_4}{168}+\frac{\widetilde{C}_5}{140}+\frac{\widetilde{C}_6}{280}\right)+\frac{K_1}{4h}\left(-\frac{\widetilde{B}_1}{15}-\frac{\widetilde{B}_2}{60}+\frac{\widetilde{B}_3}{60}-\frac{\widetilde{B}_4}{168}+\frac{\widetilde{B}_5}{140}+\right.$$

$$\left.+\frac{\widetilde{B}_6}{120}\right)+\frac{K_2}{4h}\left(-\frac{\widetilde{C}_1}{15}-\frac{\widetilde{C}_2}{60}+\frac{\widetilde{C}_3}{60}-\frac{\widetilde{C}_4}{168}+\frac{\widetilde{C}_5}{140}+\frac{\widetilde{C}_6}{120}\right)\right]+\eta^2\left[\frac{K_1^2K_2^2}{40h^2}(k_x^2+k_y^2)+\right.$$

$$\left.+\frac{\nu}{20}\frac{K_1^2K_2^2}{h^2}k_xk_y\right]-\frac{3}{2}\xi\left[\frac{1}{30}\left(k_xh\frac{K_2^2}{h^2}+k_yh\frac{K_1^2}{h^2}\right)+\frac{\nu}{30}\left(k_xh\frac{K_1^2}{h^2}+k_yh\frac{K_2^2}{h^2}\right)\right]+$$

$$+2\frac{\xi^2}{\eta^2}\left[\frac{1}{35}\left(\frac{K_2^2}{K_1^2}+\frac{K_1^2}{K_2^2}\right)+\frac{2}{105}\right]+\frac{\eta}{2\xi}\left[-\frac{k_xK_1K_2^2}{2h^2}\left(\frac{\widetilde{B}_1}{12}+\frac{\widetilde{B}_2}{40}+\frac{\widetilde{B}_3}{120}+\frac{\widetilde{B}_4}{96}+\frac{\widetilde{B}_5}{240}+\right.\right.$$

$$\left.+\frac{\widetilde{B}_6}{480}\right)-\frac{k_yK_1^2K_2}{2h^2}\left(\frac{\widetilde{C}_1}{12}+\frac{\widetilde{C}_2}{40}+\frac{\widetilde{C}_3}{120}+\frac{\widetilde{C}_4}{96}+\frac{\widetilde{C}_5}{240}+\frac{\widetilde{C}_6}{480}\right)-\frac{\nu}{4}k_x\frac{K_1^2K_2}{h^2}\left(\frac{\widetilde{C}_1}{6}+\frac{\widetilde{C}_2}{20}+\right.$$

$$\left.\left.\left.+\frac{\widetilde{C}_3}{60}+\frac{\widetilde{C}_4}{48}+\frac{\widetilde{C}_5}{120}+\frac{\widetilde{C}_6}{240}\right)-\frac{\nu}{4}k_y\frac{K_1K_2^2}{h^2}\left(\frac{\widetilde{B}_1}{6}+\frac{\widetilde{B}_2}{20}+\frac{\widetilde{B}_3}{60}+\frac{\widetilde{B}_4}{48}+\frac{\widetilde{B}_5}{120}+\frac{\widetilde{B}_6}{240}\right)\right]\right\}\tag{3.11}$$

和

$$\frac{2\xi}{6\eta^3}\left(\frac{K_2^2}{2K_1^2}+\frac{K_1^2}{2K_2^2}+\frac{1}{3}\right)+\frac{\xi}{\eta^2}\left[\frac{K_2^2}{2K_1h}\left(\frac{\widetilde{B}_2}{30}+\frac{\widetilde{B}_4}{42}+\frac{\widetilde{B}_5}{210}\right)+\frac{K_1^2}{2K_2h}\left(\frac{\widetilde{C}_2}{30}+\frac{\widetilde{C}_4}{42}+\frac{\widetilde{C}_5}{210}\right)+\right.$$

$$+\frac{\nu K_1}{4h}\left(\frac{\widetilde{B}_1}{5}+\frac{\widetilde{B}_2}{20}+\frac{\widetilde{B}_3}{60}+\frac{3\widetilde{B}_4}{168}+\frac{\widetilde{B}_5}{140}+\frac{\widetilde{B}_6}{280}\right)+\frac{\nu K_2}{4h}\left(\frac{\widetilde{C}_1}{5}+\frac{\widetilde{C}_2}{20}+\frac{\widetilde{C}_3}{60}+\frac{3\widetilde{C}_4}{168}+\frac{\widetilde{C}_5}{140}+\right.$$

$$\left.+\frac{\widetilde{C}_6}{280}\right)+\frac{K_1}{4h}\left(-\frac{\widetilde{B}_1}{15}-\frac{\widetilde{B}_2}{60}+\frac{\widetilde{B}_3}{60}-\frac{\widetilde{B}_4}{168}+\frac{\widetilde{B}_5}{140}+\frac{\widetilde{B}_6}{120}\right)+\frac{K_2}{4h}\left(-\frac{\widetilde{C}_1}{15}-\frac{\widetilde{C}_2}{60}+\frac{\widetilde{C}_3}{60}-\right.$$

$$\left.\left.-\frac{\widetilde{C}_4}{168}+\frac{\widetilde{C}_5}{140}+\frac{\widetilde{C}_6}{120}\right)\right]-2\xi\eta\left[\frac{K_1^2K_2^2}{40h^2}(k_x^2+k_y^2)+\frac{\nu}{20}\frac{K_1^2K_2^2}{h^2}k_xk_y\right]+\frac{2\xi^3}{\eta^3}\left[\frac{1}{35}\left(\frac{K_2^2}{K_1^2}+\right.\right.$$

$$\left.\left.+\frac{K_1^2}{K_2^2}\right)+\frac{2}{105}\right]+\left[k_x\frac{K_1K_2^2}{2h^2}\left(\frac{\widetilde{B}_1}{12}+\frac{\widetilde{B}_2}{40}+\frac{\widetilde{B}_3}{120}+\frac{\widetilde{B}_4}{96}+\frac{\widetilde{B}_5}{240}+\frac{\widetilde{B}_6}{480}\right)+k_y\frac{K_1^2K_2}{2h^2}\times\right.$$

$$\times\left(\frac{\widetilde{C}_1}{12}+\frac{\widetilde{C}_2}{40}+\frac{\widetilde{C}_3}{120}+\frac{\widetilde{C}_4}{96}+\frac{\widetilde{C}_5}{240}+\frac{\widetilde{C}_6}{480}\right)+\frac{\nu}{4}k_x\frac{K_1^2K_2}{h^2}\left(\frac{\widetilde{C}_1}{6}+\frac{\widetilde{C}_2}{20}+\frac{\widetilde{C}_3}{60}+\frac{\widetilde{C}_4}{48}+\frac{\widetilde{C}_5}{120}+\right.$$

$$\left.\left.+\frac{\widetilde{C}_6}{240}\right)+\frac{\nu}{4}k_y\frac{K_1K_2^2}{h^2}\left(\frac{\widetilde{B}_1}{6}+\frac{\widetilde{B}_2}{20}+\frac{\widetilde{B}_3}{60}+\frac{\widetilde{B}_4}{48}+\frac{\widetilde{B}_5}{120}+\frac{\widetilde{B}_6}{240}\right)\right]=0.\tag{3.12}$$

对失稳区橢圓半軸长的比与壳体相应曲率的比給以不同的关系式，求得的临界荷載以及模型实驗的結果，均表明橢圓拋物面双曲扁壳失稳曲面橢圓周界半軸长的比与其相应的曲率开平方成反比（詳見本文第五及第六两节）。

由此，我們取 $$K_i=\frac{h}{\sqrt{\alpha_i}},\ (i=1,\ 2)\tag{3.13}$$

式中 $$\alpha_1=k_xh,\qquad \alpha_2=k_yh.\tag{3.14}$$

将式（3.13）代入式（2.7），得

$$-\left[6+(1-\nu)\frac{\alpha_2}{\alpha_1}\right]\widetilde{B}_1+6\widetilde{B}_2+(1-\nu)\frac{\alpha_2}{\alpha_1}\widetilde{B}_3-(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_1+(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_3=$$

$$=-\left[4\left(\sqrt{\alpha_1}+\nu\frac{\alpha_2}{\sqrt{\alpha_1}}\right)\xi\eta+\frac{\xi^2}{\eta}\left(16\sqrt{\alpha_1}+8(1-\nu)\frac{\alpha_2}{\sqrt{\alpha_1}}\right)\right],$$

$$-\left[20+(1-\nu)\frac{\alpha_2}{\alpha_1}\right]\widetilde{B}_2-(1-\nu)\frac{\alpha_2}{\alpha_1}\widetilde{B}_3+20\widetilde{B}_4+2(1-\nu)\frac{\alpha_2}{\alpha_1}\widetilde{B}_5-2(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_3+$$

$$+2(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_6=4\left(\sqrt{\alpha_1}+\nu\frac{\alpha_2}{\sqrt{\alpha_1}}\right)\xi\eta+\frac{\xi^2}{\eta}\left[64\sqrt{\alpha_1}+16(1-\nu)\frac{\alpha_2}{\sqrt{\alpha_1}}\right],$$

$$\left[42+(1-\nu)\frac{\alpha_2}{\alpha_1}\right]\widetilde{B}_4+2(1-\nu)\frac{\alpha_2}{\alpha_1}\widetilde{B}_5+3(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_6=$$

$$=\frac{\xi^2}{\eta}\left[48\sqrt{\alpha_1}+8(1-\nu)\frac{\alpha_2}{\sqrt{\alpha_1}}\right],$$

$$-6\widetilde{B}_2-\left[6+6(1-\nu)\frac{\alpha_2}{\alpha_1}\right]\widetilde{B}_3+12\widetilde{B}_5+6(1-\nu)\frac{\alpha_2}{\alpha_1}\widetilde{B}_6-3(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_2-$$

$$-3(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_3+6(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_5=4\left(\sqrt{\alpha_1}+\nu\frac{\alpha_2}{\sqrt{\alpha_1}}\right)\xi\eta+\frac{\xi^2}{\eta}\left(32\sqrt{\alpha_1}+\right.$$

$$\left.+\frac{48\alpha_2}{\sqrt{\alpha_1}}-16\nu\frac{\alpha_2}{\sqrt{\alpha_1}}\right),$$

$$20\widetilde{B}_4+\left[40+12(1-\nu)\frac{\alpha_2}{\alpha_1}\right]\widetilde{B}_5+6(1-\nu)\frac{\alpha_2}{\alpha_1}\widetilde{B}_6+12(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_5+6(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_6$$

$$=\frac{\xi^2}{\eta}\left(64\sqrt{\alpha_1}+48\frac{\alpha_2}{\sqrt{\alpha_1}}-16\nu\frac{\alpha_2}{\sqrt{\alpha_1}}\right),$$

$$12\widetilde{B}_5+\left[6+15(1-\nu)\frac{\alpha_2}{\alpha_1}\right]\widetilde{B}_6+5(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_4+10(1+\nu)\sqrt{\frac{\alpha_2}{\alpha_1}}\widetilde{C}_5=$$

$$=\frac{\xi^2}{\eta}\left(16\sqrt{\alpha_1}+40\frac{\alpha_2}{\sqrt{\alpha_1}}-8\nu\frac{\alpha_2}{\sqrt{\alpha_1}}\right), \qquad (3.15)$$

$$-(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_1+(1-\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_3-\left[6+(1-\nu)\frac{\alpha_1}{\alpha_2}\right]\widetilde{C}_1+6\widetilde{C}_2+(1-\nu)\frac{\alpha_1}{\alpha_2}\widetilde{C}_3=$$

$$=-\left[4\left(\sqrt{\alpha_2}+\nu\frac{\alpha_1}{\sqrt{\alpha_2}}\right)\xi\eta+\frac{\xi^2}{\eta}\left(16\sqrt{\alpha_2}+8(1-\nu)\frac{\alpha_1}{\sqrt{\alpha_2}}\right)\right],$$

$$-2(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_3+2(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_6-\left[20+(1-\nu)\frac{\alpha_1}{\alpha_2}\right]\widetilde{C}_2-(1-\nu)\frac{\alpha_1}{\alpha_2}\widetilde{C}_3+20\widetilde{C}_4+$$

$$+2(1-\nu)\frac{\alpha_1}{\alpha_2}\widetilde{C}_5=4\left(\sqrt{\alpha_2}+\nu\frac{\alpha_1}{\sqrt{\alpha_2}}\right)\xi\eta+\frac{\xi^2}{\eta}\left[64\sqrt{\alpha_2}+16(1-\nu)\frac{\alpha_1}{\sqrt{\alpha_2}}\right],$$

$$3(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_6+\left[42+(1-\nu)\frac{\alpha_1}{\alpha_2}\right]\widetilde{C}_4+2(1-\nu)\frac{\alpha_1}{\alpha_2}\widetilde{C}_5=$$

$$=\frac{\xi^2}{\eta}\left[48\sqrt{\alpha_2}+8(1-\nu)\frac{\alpha_1}{\sqrt{\alpha_2}}\right],$$

$$-3(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_2-3(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_3+6(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_5-6\widetilde{C}_2-\left[6+6(1-\nu)\frac{\alpha_1}{\alpha_2}\right]\widetilde{C}_3$$

$$+12\widetilde{C}_5+6(1-\nu)\frac{\alpha_1}{\alpha_2}\widetilde{C}_6=4\left(\sqrt{\alpha_2}+\nu\frac{\alpha_1}{\sqrt{\alpha_2}}\right)\xi\eta+\frac{\xi^2}{\eta}\left(32\sqrt{\alpha_2}+48\frac{\alpha_1}{\sqrt{\alpha_2}}-16\nu\frac{\alpha_1}{\sqrt{\alpha_2}}\right),$$

$$12(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_5+6(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_6+20\widetilde{C}_4+\left[40+12(1-\nu)\frac{\alpha_1}{\alpha_2}\right]\widetilde{C}_5+6(1-\nu)\frac{\alpha_1}{\alpha_2}\widetilde{C}_6$$

$$=\frac{\xi^2}{\eta}\left(64\sqrt{\alpha_2}+48\frac{\alpha_1}{\sqrt{\alpha_2}}-16\nu\frac{\alpha_1}{\sqrt{\alpha_2}}\right),$$

$$5(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_4+10(1+\nu)\sqrt{\frac{\alpha_1}{\alpha_2}}\widetilde{B}_5+12\widetilde{C}_5+\left[6+15(1-\nu)\frac{\alpha_1}{\alpha_2}\right]\widetilde{C}_6=$$
$$=\frac{\xi^2}{\eta}\left(16\sqrt{\alpha_2}+40\frac{\alpha_1}{\sqrt{\alpha_2}}-8\nu\frac{\alpha_1}{\sqrt{\alpha_2}}\right).$$

式中

$$\widetilde{B}_i=\begin{cases}b_i\xi\eta+b_i'\frac{\xi^2}{\eta}, & (i=1,\ 2,\ 3)\\ b_i'\frac{\xi^2}{\eta}; & (i=4,\ 5,\ 6)\end{cases} \tag{3.16}$$

$$\widetilde{C}_i=\begin{cases}c_i\xi\eta+c_i'\frac{\xi^2}{\eta}, & (i=1,\ 2,\ 3)\\ c_i'\frac{\xi^2}{\eta}. & (i=4,\ 5,\ 6)\end{cases} \tag{3.17}$$

对于等曲率之双曲扁壳（$k_x=k_y$，$a=b$，$\alpha_1=\alpha_2$）由式（3.15）可得

$$\widetilde{B}_i=\widetilde{C}_i,\quad (i=1,\ 2,\ \cdots\cdots,\ 6) \tag{3.18}$$

壳体失稳曲面周界的水平投影退化为一个圆，式（3.15）簡化为

$$\left.\begin{aligned}
&-8\widetilde{B}_1+6\widetilde{B}_2+2\widetilde{B}_3=-\left[4\sqrt{\alpha}(1+\nu)\xi\eta+\frac{\xi^2}{\eta}(24-8\nu)\sqrt{\alpha}\right],\\
&-(21-\nu)\widetilde{B}_2-(3+\nu)\widetilde{B}_3+20\widetilde{B}_4+2(1-\nu)\widetilde{B}_5+2(1+\nu)\widetilde{B}_6=\\
&\qquad=4\sqrt{\alpha}(1+\nu)\xi\eta+\frac{\xi^2}{\eta}\left[64\sqrt{\alpha}+16(1-\nu)\sqrt{\alpha}\right],\\
&(43-\nu)\widetilde{B}_4+2(1-\nu)\widetilde{B}_5+3(1+\nu)\widetilde{B}_6=\frac{\xi^2}{\eta}\left[56\sqrt{\alpha}-8\nu\sqrt{\alpha}\right],\\
&-(9+3\nu)\widetilde{B}_2-(15-3\nu)\widetilde{B}_3+(18+6\nu)\widetilde{B}_5+6(1-\nu)\widetilde{B}_6=\\
&\qquad=4\sqrt{\alpha}(1+\nu)\xi\eta+\frac{\xi^2}{\eta}(80\sqrt{\alpha}-16\nu\sqrt{\alpha}),\\
&20\widetilde{B}_4+64\widetilde{B}_5+12\widetilde{B}_6=\frac{\xi^2}{\eta}(112\sqrt{\alpha}-16\nu\sqrt{\alpha}),\\
&5(1+\nu)\widetilde{B}_4+(22+10\nu)\widetilde{B}_5+(21-15\nu)\widetilde{B}_6=\frac{\xi^2}{\eta}(56\sqrt{\alpha}-8\nu\sqrt{\alpha}).
\end{aligned}\right\} \tag{3.19}$$

式中

$$\alpha=\alpha_1=\alpha_2=kh. \tag{3.20}$$

不难求得式（3.19）的解为

$$\left.\begin{aligned}
&\widetilde{B}_1=\frac{(1+\nu)\sqrt{\alpha}}{3}\xi\eta+\frac{5-3\nu}{6}\sqrt{\alpha}\frac{\xi^2}{\eta},\\
&\widetilde{B}_2=\widetilde{B}_3=-\frac{1+\nu}{6}\sqrt{\alpha}\xi\eta-\frac{13-3\nu}{6}\sqrt{\alpha}\frac{\xi^2}{\eta},\\
&\widetilde{B}_4=\widetilde{B}_5=\widetilde{B}_6=\frac{7-\nu}{6}\sqrt{\alpha}\frac{\xi^2}{\eta}.
\end{aligned}\right\} \tag{3.21}$$

将 $\widetilde{B}_i$ 与 $\widetilde{C}_i$ 的解及式（3.13）的关系式代入式（3.11）及（3.12），經整理后得

$$q^*=\left(\psi_1\frac{1}{\eta^2}+\psi_2\frac{\xi^2}{\eta^2}+\psi_3\xi+\psi_4\eta^2\right)k_xk_yh^2, \tag{3.22}$$

其中

$$\psi_1=\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}+\frac{2}{3}, \tag{3.23}$$

$$\psi_2=\frac{3}{2}\left[\frac{\sqrt{\alpha_1}}{\alpha_2}\left(\frac{2b_2'}{15}+\frac{2b_4'}{21}+\frac{2b_5'}{105}\right)+\frac{\sqrt{\alpha_2}}{\alpha_1}\left(\frac{2c_2'}{15}+\frac{2c_4'}{21}+\frac{2c_5'}{105}\right)+\frac{\nu}{\sqrt{\alpha_1}}\left(\frac{2b_1'}{5}+\frac{b_2'}{10}+\right.\right.$$
$$+\frac{b_3'}{30}+\frac{3b_4'}{84}+\frac{b_5'}{70}+\frac{b_6'}{140}\Big)+\frac{\nu}{\sqrt{\alpha_2}}\left(\frac{2c_1'}{5}+\frac{c_2'}{10}+\frac{c_3'}{30}+\frac{3c_4'}{84}+\frac{c_5'}{70}+\frac{c_6'}{140}\right)+\frac{16}{35}\times$$
$$\times\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)+\frac{32}{105}+\frac{1}{\sqrt{\alpha_1}}\left(-\frac{2b_1'}{15}-\frac{b_2'}{30}+\frac{b_3'}{30}-\frac{b_4'}{84}+\frac{b_5'}{70}+\frac{b_6'}{60}\right)+\frac{1}{\sqrt{\alpha_2}}\times$$
$$\left.\times\left(-\frac{2c_1'}{15}-\frac{c_2'}{30}+\frac{c_3'}{30}-\frac{c_4'}{84}+\frac{c_5'}{70}+\frac{c_6'}{60}\right)\right], \tag{3.24}$$

$$\psi_3=\frac{3}{2}\left[\frac{2}{15}\left(b_2\frac{\sqrt{\alpha_1}}{\alpha_2}+c_2\frac{\sqrt{\alpha_2}}{\alpha_1}\right)-\frac{2}{5}\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)-\frac{4}{5}\nu+\frac{\nu}{\sqrt{\alpha_1}}\left(\frac{2b_1}{5}+\frac{b_2}{10}+\right.\right.$$
$$+\frac{b_3}{30}\Big)+\frac{\nu}{\sqrt{\alpha_2}}\left(\frac{2c_1}{5}+\frac{c_2}{10}+\frac{c_3}{30}\right)-\frac{1}{\sqrt{\alpha_1}}\left(\frac{2b_1}{15}+\frac{b_2}{30}-\frac{b_3}{30}\right)-\frac{1}{\sqrt{\alpha_2}}\left(\frac{2c_1}{15}+\frac{c_2}{30}-\right.$$
$$-\frac{c_3}{30}\Big)-2\frac{\sqrt{\alpha_1}}{\alpha_2}\left(\frac{b_1'}{12}+\frac{b_2'}{40}+\frac{b_3'}{120}+\frac{b_4'}{96}+\frac{b_5'}{240}+\frac{b_6'}{480}\right)-2\frac{\sqrt{\alpha_2}}{\alpha_1}\left(\frac{c_1'}{12}+\frac{c_2'}{40}+\right.$$
$$+\frac{c_3'}{120}+\frac{c_4'}{96}+\frac{c_5'}{240}+\frac{c_6'}{480}\Big)-\frac{\nu}{\sqrt{\alpha_2}}\left(\frac{c_1'}{6}+\frac{c_2'}{20}+\frac{c_3'}{60}+\frac{c_4'}{48}+\frac{c_5'}{120}+\frac{c_6'}{240}\right)-\frac{\nu}{\sqrt{\alpha_1}}\times$$
$$\left.\times\left(\frac{b_1'}{6}+\frac{b_2'}{20}+\frac{b_3'}{60}+\frac{b_4'}{48}+\frac{b_5'}{120}+\frac{b_6'}{240}\right)\right], \tag{3.25}$$

$$\psi_4=\frac{3}{2}\left[\frac{1}{5}\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)+\frac{2}{5}\nu-\frac{2\sqrt{\alpha_1}}{\alpha_2}\left(\frac{1}{12}b_1+\frac{1}{40}b_2+\frac{1}{120}b_3\right)-\frac{2\sqrt{\alpha_2}}{\alpha_1}\left(\frac{c_1}{12}+\right.\right.$$
$$\left.+\frac{c_2}{40}+\frac{c_3}{120}\Big)-\frac{\nu}{\sqrt{\alpha_2}}\left(\frac{c_1}{6}+\frac{c_2}{20}+\frac{c_3}{60}\right)-\frac{\nu}{\sqrt{\alpha_1}}\left(\frac{b_1}{6}+\frac{b_2}{20}+\frac{b_3}{60}\right)\right]; \tag{3.26}$$

又得

$$-\Phi_1\eta^4+\Phi_2\xi^2+\Phi_3+\Phi_4\xi\eta^2=0, \tag{3.27}$$

式中

$$\Phi_1=-\left[\frac{\sqrt{\alpha_1}}{2\alpha_2}\left(\frac{b_1}{12}+\frac{b_2}{40}+\frac{b_3}{120}\right)+\frac{\sqrt{\alpha_2}}{2\alpha_1}\left(\frac{c_1}{12}+\frac{c_2}{40}+\frac{c_3}{120}\right)-\frac{1}{20}\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)+\right.$$
$$\left.+\frac{\nu}{4\sqrt{\alpha_2}}\left(\frac{c_1}{6}+\frac{c_2}{20}+\frac{c_3}{60}\right)+\frac{\nu}{4\sqrt{\alpha_1}}\left(\frac{b_1}{6}+\frac{b_2}{20}+\frac{b_3}{60}\right)-\frac{\nu}{10}\right], \tag{3.28}$$

$$\Phi_2=\left[\frac{\sqrt{\alpha_1}}{2\alpha_2}\left(\frac{b_2'}{30}+\frac{b_4'}{42}+\frac{b_5'}{210}\right)+\frac{\sqrt{\alpha_2}}{2\alpha_1}\left(\frac{c_2'}{30}+\frac{c_4'}{42}+\frac{c_5'}{210}\right)+\frac{2}{35}\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)+\frac{4}{105}+\right.$$
$$+\frac{1}{4\sqrt{\alpha_1}}\left(-\frac{b_1'}{15}-\frac{b_2'}{60}+\frac{b_3'}{60}-\frac{b_4'}{168}+\frac{b_5'}{140}+\frac{b_6'}{120}\right)+\frac{1}{4\sqrt{\alpha_2}}\left(-\frac{c_1'}{15}-\frac{c_2'}{60}+\frac{c_3'}{60}-\right.$$
$$-\frac{c_4'}{168}+\frac{c_5'}{140}+\frac{c_6'}{120}\Big)+\frac{\nu}{4\sqrt{\alpha_1}}\left(\frac{b_1'}{5}+\frac{b_2'}{20}+\frac{b_3'}{60}+\frac{b_4'}{168}+\frac{b_5'}{140}+\frac{b_6'}{280}\right)+\frac{\nu}{4\sqrt{\alpha_2}}\times$$
$$\left.\times\left(\frac{c_1'}{5}+\frac{c_2'}{20}+\frac{c_3'}{60}+\frac{c_4'}{168}+\frac{c_5'}{140}+\frac{c_6'}{280}\right)\right], \tag{3.29}$$

$$\Phi_3=\frac{1}{3}\left(\frac{\alpha_1}{2\alpha_2}+\frac{\alpha_2}{2\alpha_1}+\frac{1}{3}\right), \tag{3.30}$$

$$\Phi_4=\frac{1}{60}\left(b_2\frac{\sqrt{\alpha_1}}{\alpha_2}+c_2\frac{\sqrt{\alpha_2}}{\alpha_1}\right)+\frac{1}{4\sqrt{\alpha_1}}\left(-\frac{b_1}{15}-\frac{b_2}{60}+\frac{b_3}{60}\right)+\frac{1}{4\sqrt{\alpha_2}}\left(-\frac{c_1}{15}-\frac{c_2}{60}+\right.$$

$$+\frac{c_3}{60}\Big)+\frac{\nu}{4\sqrt{\alpha_1}}\Big(\frac{b_1}{5}+\frac{b_2}{20}+\frac{b_3}{60}\Big)+\frac{\nu}{4\sqrt{\alpha_2}}\Big(\frac{c_1}{5}+\frac{c_2}{20}+\frac{c_3}{60}\Big)+\frac{\sqrt{\alpha_1}}{2\alpha_2}\Big(\frac{b'_1}{12}+$$

$$+\frac{b'_2}{40}+\frac{b'_3}{120}+\frac{b'_4}{96}+\frac{b'_5}{240}+\frac{b'_6}{480}\Big)+\frac{\sqrt{\alpha_2}}{2\alpha_1}\Big(\frac{c'_1}{12}+\frac{c'_2}{40}+\frac{c'_3}{120}+\frac{c'_4}{96}+\frac{c'_5}{240}+\frac{c'_6}{480}\Big)+$$

$$+\frac{\nu}{4\sqrt{\alpha_2}}\Big(\frac{c'_1}{6}+\frac{c'_2}{20}+\frac{c'_3}{60}+\frac{c'_4}{48}+\frac{c'_5}{120}+\frac{c'_6}{240}\Big)+\frac{\nu}{4\sqrt{\alpha_1}}\Big(\frac{b'_1}{6}+\frac{b'_2}{20}+\frac{b'_3}{60}+$$

$$+\frac{b'_4}{48}+\frac{b'_5}{120}+\frac{b'_6}{240}\Big). \qquad (3.31a)$$

显然，ψ_i与Φ_i均为仅与曲率比有关的系数。

解式（3.15），将求得的b_i，b'_i，c_i及c'_i代入式（3.31a）得

$$\Phi_4=0, \qquad (3.31b)$$

式（3.27）可簡化为

$$\eta^2=\sqrt{\frac{\Phi_2}{\Phi_1}\xi^2+\frac{\Phi_3}{\Phi_1}}. \qquad (3.32)$$

将上式代入（3.22），得

$$q^*=\left(\frac{\psi_1+\psi_2\xi^2}{\sqrt{\frac{\Phi_2}{\Phi_1}\xi^2+\frac{\Phi_3}{\Phi_1}}}+\psi_3\xi+\psi_4\sqrt{\frac{\Phi_2}{\Phi_1}\xi^2+\frac{\Phi_3}{\Phi_1}}\right)k_xk_yh^2. \qquad (3.33)$$

对等曲率之双曲扁壳（$\alpha_1=\alpha_2=\alpha$），系数ψ_i及Φ_i分别簡化为

$$\left.\begin{aligned}&\psi_1=\frac{8}{3}, &&\psi_3=-\frac{3}{2}-\nu+\frac{1}{2}\nu^2,\\ &\psi_2=\frac{23}{21}+\frac{2}{3}\nu-\frac{3}{7}\nu^2, &&\psi_4=\frac{7}{15}+\frac{1}{3}\nu-\frac{2}{15}\nu^2,\end{aligned}\right\} \qquad (3.34)$$

及

$$\Phi_1=\frac{1}{4}\Big(\frac{14}{45}+\frac{2}{9}\nu-\frac{4}{45}\nu^2\Big),\ \Phi_2=\frac{1}{4}\Big(\frac{23}{63}+\frac{2}{9}\nu-\frac{1}{7}\nu^2\Big),\ \Phi_3=\frac{4}{9}. \qquad (3.35)$$

式（3.32）与（3.33）分别簡化为

$$\eta^2=\sqrt{\frac{\frac{16}{9}+\Big(\frac{23}{63}+\frac{2}{9}\nu-\frac{1}{7}\nu^2\Big)\xi^2}{\frac{14}{45}+\frac{2}{9}\nu-\frac{4}{45}\nu^2}}, \qquad (3.36)$$

$$q^*=\Big[\frac{8}{3}\ \frac{1}{\eta^2}+\Big(\frac{23}{21}+\frac{2}{3}\nu-\frac{3}{7}\nu^2\Big)\frac{\xi^2}{\eta^2}+\Big(-\frac{3}{2}-\nu+\frac{1}{2}\nu^2\Big)\xi+$$

$$+\Big(\frac{7}{15}+\frac{1}{3}\nu-\frac{2}{15}\nu^2\Big)\eta^2\Big]k^2h^2, \qquad (3.37)$$

这与球壳非线性稳定問題的方程一样。

由$\frac{\partial q^*}{\partial \xi}=0$的条件，可求得壳体的临界荷載$q_{cr}=Eq^*_{min}$，考虑到$q^*-\xi$关系式較为复杂，实用上可采用作图法进行計算。

四、用能量法計算椭圆抛物面双曲扁壳的稳定性

扁壳丧失稳定时，总势能I包括以下四个部分：

1.弯曲应变能

$$V_1=\frac{D}{2}\iint_F\Big[\Big(\frac{\partial^2 w}{\partial x^2}\Big)^2+\Big(\frac{\partial^2 w}{\partial y^2}\Big)^2+2\nu\frac{\partial^2 w}{\partial x^2}\ \frac{\partial^2 w}{\partial y^2}+2(1-\nu)\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2\Big]dxdy, \qquad (4.1)$$

2.薄膜应变能

$$V_2=\frac{Eh}{2(1-\nu^2)}\iint_F\left\{\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2+\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]^2+\right.$$

$$+2\nu\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{1-\nu}{2}\left(\frac{\partial u}{\partial y}+\right.$$

$$\left.\left.+\frac{\partial v}{\partial x}-2k_{xy}w+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)^2\right\}dxdy, \tag{4.2}$$

3.失稳前壳体薄膜力在失稳过程中所作的功

$$V_3=-\frac{q}{2k_x}\iint_F\left[\frac{\partial u}{\partial x}-k_xw+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dxdy-$$

$$-\frac{q}{2k_y}\iint_F\left[\frac{\partial v}{\partial y}-k_yw+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]dxdy, \tag{4.3}$$

4.外荷載所作的功

$$V_4=\iint_F qwdxdy. \tag{4.4}$$

式（4.1）～（4.4）的积分范围遍及壳面失稳区域，由F表示其面积。

按势能的定义得

$$I=V_1+V_2+V_3-V_4. \tag{4.5}$$

将失稳区位移函数（2.1），（2.3），（2.4）代入式（4.5）进行积分，将积分結果无量綱化，得无量綱总势能$\widetilde{I}$的公式为

$$\widetilde{I}=\frac{1-\nu^2}{4\pi h^3E}I=\frac{\xi^2}{\eta^2}\left[\frac{h^2}{6}\left(\frac{K_2}{2K_1^3}+\frac{K_1}{2K_2^3}+\frac{1}{3K_1K_2}\right)\right]+\frac{\xi^2}{\eta}\left[\frac{K_2h}{2K_1^2}\left(\frac{\widetilde{B}_2}{30}+\frac{\widetilde{B}_4}{42}+\frac{\widetilde{B}_5}{210}\right)+\frac{K_1h}{2K_2^2}\times\right.$$

$$\times\left(\frac{\widetilde{C}_2}{30}+\frac{\widetilde{C}_4}{42}+\frac{\widetilde{C}_5}{210}\right)+\frac{\nu h}{4K_2}\left(\frac{\widetilde{B}_1}{5}+\frac{\widetilde{B}_2}{20}+\frac{\widetilde{B}_3}{60}+\frac{3B_4}{168}+\frac{\widetilde{B}_5}{140}+\frac{\widetilde{B}_6}{280}\right)+\frac{\nu h}{4K_1}\left(\frac{\widetilde{C}_1}{5}+\frac{\widetilde{C}_2}{20}\right.$$

$$\left.+\frac{\widetilde{C}_3}{60}+\frac{3\widetilde{C}_4}{168}+\frac{\widetilde{C}_5}{140}+\frac{\widetilde{C}_6}{280}\right)+\frac{h}{4K_2}\left(-\frac{\widetilde{B}_1}{15}-\frac{\widetilde{B}_2}{60}+\frac{\widetilde{B}_3}{60}-\frac{\widetilde{B}_4}{168}+\frac{\widetilde{B}_5}{140}+\frac{\widetilde{B}_6}{120}\right)+\frac{h}{4K_1}\left(-\right.$$

$$\left.\left.-\frac{\widetilde{C}_1}{15}-\frac{\widetilde{C}_2}{60}+\frac{\widetilde{C}_3}{60}-\frac{\widetilde{C}_4}{168}+\frac{\widetilde{C}_5}{140}+\frac{\widetilde{C}_6}{120}\right)\right]+\xi^2\eta^2\left[\frac{1}{40}K_1K_2(k_x^2+k_y^2)+\frac{\nu}{20}K_1K_2k_xk_y\right]-$$

$$-\frac{1}{24}q^*\xi^2\left(\frac{K_2}{K_1k_xh}+\frac{K_1}{K_2k_yh}\right)-\xi^3\left[\frac{1}{30}\left(k_xh\frac{K_2}{K_1}+k_yh\frac{K_1}{K_2}\right)+\frac{\nu}{30}\left(k_xh\frac{K_1}{K_2}+\right.\right.$$

$$\left.\left.+k_yh\frac{K_2}{K_1}\right)\right]+\frac{\xi^4}{\eta^2}\left[\frac{h^2}{35}\left(\frac{K_2}{K_1^3}+\frac{K_1}{K_2^3}\right)+\frac{2h^2}{105K_1K_2}\right]+\xi\eta\left[-\frac{k_xK_2}{2}\left(\frac{\widetilde{B}_1}{12}+\frac{\widetilde{B}_2}{40}+\right.\right.$$

$$\left.+\frac{\widetilde{B}_3}{120}+\frac{\widetilde{B}_4}{96}+\frac{\widetilde{B}_5}{240}+\frac{\widetilde{B}_6}{480}\right)-\frac{k_yK_1}{2}\left(\frac{\widetilde{C}_1}{12}+\frac{\widetilde{C}_2}{40}+\frac{\widetilde{C}_3}{120}+\frac{\widetilde{C}_4}{96}+\frac{\widetilde{C}_5}{240}+\frac{\widetilde{C}_6}{480}\right)-$$

$$-\frac{\nu}{4}k_xK_1\left(\frac{\widetilde{C}_1}{6}+\frac{\widetilde{C}_2}{20}+\frac{\widetilde{C}_3}{60}+\frac{\widetilde{C}_4}{48}+\frac{\widetilde{C}_5}{120}+\frac{\widetilde{C}_6}{240}\right)-\frac{\nu}{4}k_yK_2\left(\frac{\widetilde{B}_1}{6}+\frac{\widetilde{B}_2}{20}+\frac{\widetilde{B}_3}{60}+\right.$$

$$\left.\left.+\frac{\widetilde{B}_4}{48}+\frac{\widetilde{B}_5}{120}+\frac{\widetilde{B}_6}{240}\right)\right]+I_n(\widetilde{B}_i,\ \widetilde{C}_i). \tag{4.6}$$

式中 $I_n(\widetilde{B}_i,\ \widetilde{C}_i)$为仅与$\widetilde{B}_i$，$\widetilde{C}_i$項有关的势能，为节省篇幅，其表达式从略。

势能为极小时，有

$$\frac{\partial\widetilde{I}}{\partial\xi}=0,\qquad \frac{\partial\widetilde{I}}{\partial\eta}=0, \tag{4.7}$$

$$\frac{\partial\widetilde{I}}{\partial\widetilde{B}_i}=0,\qquad \frac{\partial\widetilde{I}}{\partial\widetilde{C}_i}=0.\qquad (i=1,\ 2,\ \cdots\cdots,\ 6) \tag{4.8}$$

式（4.7）給出方程

$$q^*=\left(\psi_1\frac{1}{\eta^2}+\psi_2\frac{\xi^2}{\eta^2}+\psi_3\xi+\psi_4\eta^2\right)k_xk_yh^2, \tag{4.9}$$

$$-\Phi_1\eta^4+\Phi_2\xi^2+\Phi_3=0, \tag{4.10}$$

它們与用广义伽辽金变分方程求得的結果一致。

式（4.8）給出$\widetilde{B}_i$、$\widetilde{C}_i$之线性代数方程組，不难証明，它們只是式（3.15）的线性組合，故二者的解相同。

五、双曲扁壳临界荷載的計算及实用計算公式

給定不同的系数比K_1/K_2，由式（3.11）及（3.12）可求出不同的临界荷載。令

$$\frac{K_1}{K_2}=\left(\frac{k_y}{k_x}\right)^n,$$

給n以一系列不同的值（$0<n\leqslant 1$），分别由式（3.11）及（3.12）求出相应的临界荷载q^*_{cr}。結果表明，$n=\frac{1}{2}$时的壳体临界荷載取极小値，这表明式（3.13）的关系是正确的，卽壳体失稳区域半軸长之比系与相应的曲率开平方成反比。图 5.1 列出了不同n值时的$q^*-\xi$曲线；图5.2为$q^*_{min}-n$曲线。

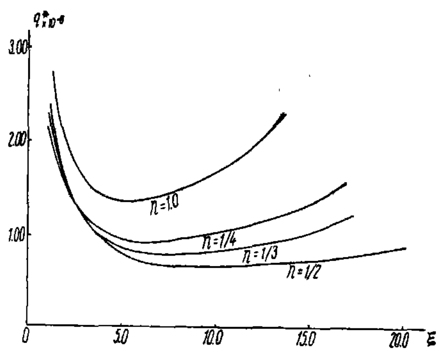


图 5.1

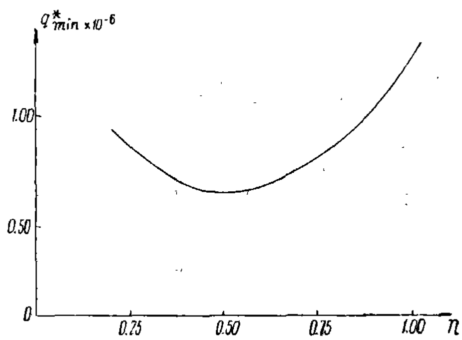


图 5.2

下面我們来推导椭圆抛物面双曲扁壳在均匀外压作用下临界荷載的实用公式。

因ψ_i与Φ_i均只与曲率比有关，由式（3.33）可知，壳体临界荷载公式可表示为

$$q_{cr}=C_0\left(\frac{k_x}{k_y}\right)Ek_xk_yh^2. \tag{5.1}$$

式中 $C_0\left(\frac{k_x}{k_y}\right)$为只与曲率比有关的系数。

实用上，不等曲率双曲扁壳的曲率比一般符合

$$\frac{k_x}{k_y}\leqslant 2, \tag{5.2}$$

我們取 $(k_x/k_y)=1.0$，1.25，1.50，1.75，2.0 等五种情况，計算壳体的临界荷載。

为簡单計，取壳体的泊松系数 ν 为零，这对鋼筋混凝土壳体是足够精确的。

确定曲率比值后，由方程（3.15）可解出 $\tilde{B}_i$ 与 $\tilde{C}_i$，但为了方便，我們給定 α_1 及 α_2 以具体数值进行运算，利用式（3.16）及（3.17）求得 b_i，b'_i，c_i，c'_i 值，代入式（3.33）及（3.37），得到不同曲率比时 $\eta^2-\xi$ 及 $q^*-\xi$ 的关系式：

（1）$\dfrac{k_x}{k_y}=1.0$ 时，

$$\left.\begin{aligned}\eta^2&=\sqrt{1.1735\xi^2+5.7134},\\ q^*&=\Bigg(\frac{2.66667+1.0952\xi^2}{\sqrt{1.1735\xi^2+5.7134}}+\\&\quad+0.46667\sqrt{1.1735\xi^2+5.7134}-\\&\quad-1.5\xi\Bigg)k^2h^2;\end{aligned}\right\}\quad(5.3)$$

（2）$\dfrac{k_x}{k_y}=1.25$ 时，

$$\left.\begin{aligned}\eta^2&=\sqrt{1.17178\xi^2+5.69679},\\ q^*&=\Bigg(\frac{2.71667+1.11759\xi^2}{\sqrt{1.17178\xi^2+5.69679}}+\\&\quad+0.47688\sqrt{1.17178\xi^2+5.69679}\\&\quad-1.53188\xi\Bigg)k_xk_yh^2;\end{aligned}\right\}\quad(5.4)$$

（3）$\dfrac{k_x}{k_y}=1.50$ 时，

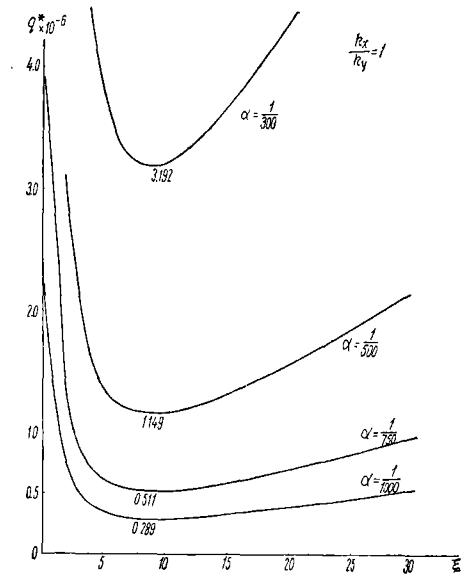


图 5.3 $\dfrac{k_x}{k_y}=1$

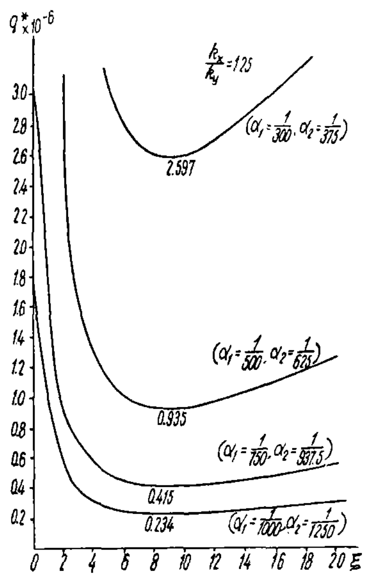


图 5.4 $\dfrac{k_x}{k_y}=1.25$

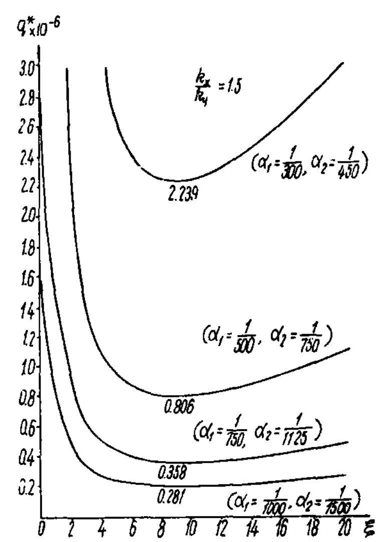


图 5.5 $\dfrac{k_x}{k_y}=1.5$

$$\left.\begin{aligned}\eta^2&=\sqrt{1.16813\xi^2+5.65857},\\ q^*&=\left(\frac{2.83333+1.16980\xi^2}{\sqrt{1.16813\xi^2+5.65857}}+0.50072\sqrt{1.16813\xi^2+5.65857}-1.60632\xi\right)k_xk_yh^2;\end{aligned}\right\}\quad(5.5)$$

（4）$\frac{k_x}{k_y}=1.75$，时，

$$\left.\begin{aligned}\eta^2&=\sqrt{1.16383\xi^2+5.61283},\\ q^*&=\left(\frac{2.98809+1.23917\xi^2}{\sqrt{1.16383\xi^2+5.61283}}+0.53237\sqrt{1.16383\xi^2+5.61283}-1.70522\xi\right)k_xk_yh^2.\end{aligned}\right\}\quad(5.6)$$

（5）$\frac{k_x}{k_y}=2.00$时，

$$\left.\begin{aligned}\eta^2&=\sqrt{1.15951\xi^2+5.56599},\\ q^*&=\left(\frac{3.16666+1.31936\xi^2}{\sqrt{1.15951\xi^2+5.56599}}+0.56893\sqrt{1.15951\xi^2+5.56599}-1.81947\xi\right)k_xk_yh^2.\end{aligned}\right\}\quad(5.7)$$

作$q^*-\xi$曲线，得到图5.3～5.7，由图可看出$\xi\approx9$时，q^*取极小值。

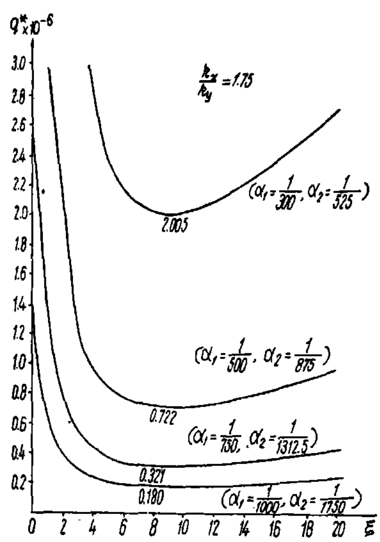


图 5.6 $\frac{k_x}{k_y}=1.75$

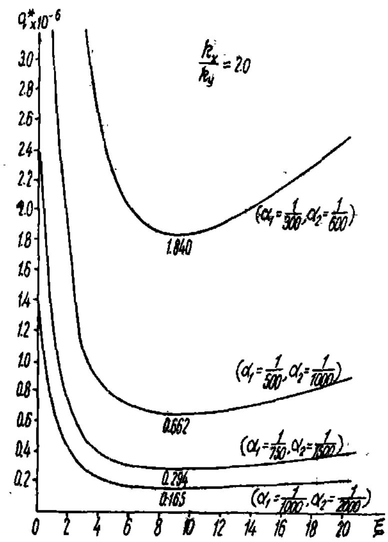


图 5.7 $\frac{k_x}{k_y}=2.0$

經过換算求得不同曲率比时双曲扁壳的临界荷載系数C_0之值为

$k_x/k_y=1.0$，	$C_0=0.288$；	$k_x/k_y=1.75$，	$C_0=0.315$；
$k_x/k_y=1.25$，	$C_0=0.292$；	$k_x/k_y=2.0$，	$C_0=0.331$．
$k_x/k_y=1.50$，	$C_0=0.302$；		

利用最小二乘法原理可得C_0的近似表达式为

$$C_0=0.288+0.013\lambda+0.03\lambda^2,\quad(5.8)$$

式中

$$\lambda=\frac{k_x}{k_y}-1.\quad(0\leqslant\lambda\leqslant1)\quad(5.9)$$

将式（5.8）代入式（5.1）得双曲扁壳临界荷載的实用公式

$$q_{cr}=(0.288+0.013\lambda+0.03\lambda^2)Ek_xk_yh^2.\quad(5.10)$$

德国H.Schmidt[10]通过八个鋁制双曲扁壳模型稳定实験求得的临界荷載系数为0.32，这相当于$\frac{k_x}{k_y}=1.85$时，由本文式（5.10）給出的結果。

考虑到壳面原始缺陷的存在，鋼筋混凝土材料及壳面厚度的非均匀性，荷載长期作用及計算簡图与实际情況尚存在一定差異等因素均可能导致壳体临界荷載的降低，在引用式（5.10）时尚需引入适当的安全系数。

苏联鋼筋混凝土薄壁空間結构設計計算规程[11]对于球壳的稳定公式规定为$q_{cr}=0.05E\frac{h^2}{R^2}$与Вольмир求得的理論解答$q_{cr}=0.288E\frac{h^2}{R^2}$[5]相比較*，安全系数約为5.7。根据文献〔12〕的解释，主要是考虑混凝土材料的蠕变与非匀质性。不等曲率双曲扁壳的稳定計算公式，文献〔11〕尙未解决，沒有相应的规定条文。

在我国"鋼筋混凝土薄壳頂盖及楼盖結构設計計算暫行规程"审訂会議上，与会专家們对不等曲率双曲扁壳的稳定公式进行了反复的研究与討論。根据上述研究結果，結合我国已建工程的实践經驗，并考虑到鋼筋混凝土的蠕变与非匀质性，以及壳面原始缺陷等因素的影响，建議双曲扁壳临界荷載的公式为

$$q_{cr}=0.06Ek_xk_yh^2。\qquad (5.11)$$

式（5.10）中的λ項已被略去，因其影响至多只有15%，而且略去后可使結果更偏于安全方面。

六、模 型 实 験

模型分A、B两組，共六个試件。A組为不等曲率双曲扁壳，試件四个，$k_x/k_y=1.5$，$k_xh=\frac{1}{300}$；B組为等曲率双曲扁壳，試件两个，$k_x=k_y$，$k_xh=1/350$；模型用0.1cm厚黃銅板制成，壳体两个方向的边长均为30cm。

模型壳板最好在冲床上用冲压法制造，可保証壳面几何尺寸准确，厚度均匀。但由于設备所限，上述六个模型是用手工制成的，除模型A_4外，壳面基本上光滑、无明显錘打痕迹，几何尺寸一般符合設計要求，仅模型A_8的曲率略为偏大。試験后用卡具对一部分模型壳板的厚度进行了测量，厚度基本上是均匀的，正負誤差都在5%以內。

壳板焊牢在加荷設备盖板的实腹鋼边梁上，其边界接近为嵌固，这样处理主要是为了好作試験。对于局部失稳問題，壳体开始失稳瞬間的失稳区域較小，因此，其实际周边边界的影响是不大的。

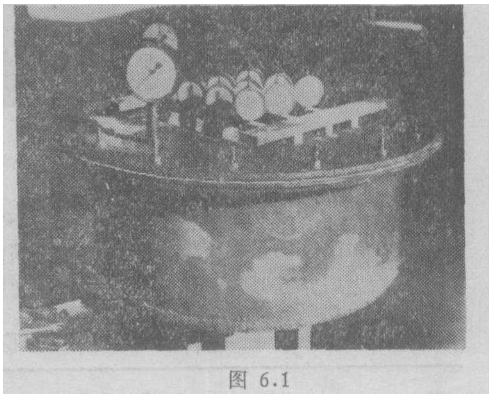


图 6.1

壳体加荷設备（图6.1）为鋼板制成的圓形容器，直径90cm、高80cm，頂部有一圈圓形垫板，上复焊有模型壳板的盖板，使壳体凸面朝里，两者間垫以硬橡皮垫圈，擰紧螺栓，造成容器的密封状态。容器內灌滿机油，通过油泵均匀地对壳面施加压力，压力值由两个压力表讀出。由于用水作加压的介质，在加荷时将引起振动，我們未予采用。

实験时对壳面逐漸加大压力，壳面位移相应增大，可用事先設置的百分表讀出（图6.1），当压力临近失稳荷載时，将百分表卸去，然后继續增大压力，直至壳面突然发生面部凹陷，同时伴随以"嘭"的声响，此时即认为壳体丧失了稳定，壳体发生凹陷瞬間讀得的压力值即为壳体的临界荷載q_{cr}。

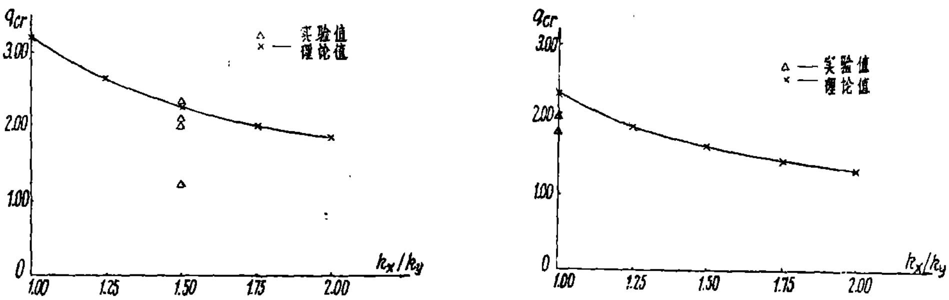
表1及图6.2是两組模型的临界荷載实験值与理論值的比較。其中，模型A_4壳面存在着肉眼明显可見的

* 略去泊松比ν时的結果。

模型临界荷載（kg/cm^2） **表 1**

試件 / 項目	A_1	A_2	A_3	A_4	B_1	B_2
实驗值	2.00	2.10	2.30	1.20	2.05	1.85
理論值	2.24	2.24	2.24	2.24	2.34	2.34

局部凹凸現象，实驗时，临界荷載显著降低。計算理論临界荷載时，弹性模量用标准值$E=1\times10^6 kg/cm^2$，測定結果与标准值接近。



（a） $k_xh=\frac{1}{300}$ （b） $k_xh=\frac{1}{350}$

图 6.2

图 6.3

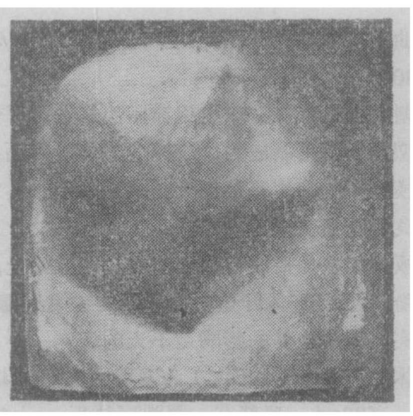


图 6.4

图6.3及图6.4分别为模型A_1及B_1壳面失稳后之凹陷状态。

表 2 給出了两組模型失稳曲面周界半軸长之比及其与理論結果的比較。实驗清楚地表明，A組模型之失稳曲面之周界近似为一椭圓，而B組模型失稳曲面之周界近似是一个圓。

模型失稳曲面周界半軸长比值a/b **表 2**

試件 / 項目	A_1	A_2	A_3	B_1	B_2
实驗值	$\frac{1}{1.25}$	$\frac{1}{1.20}$	$\frac{1}{1.21}$	1.03	1.03
理論值	$\frac{1}{1.23}$	$\frac{1}{1.23}$	$\frac{1}{1.23}$	1.00	1.00

注：模型A_4因原始缺陷影响，其失稳曲面呈不規則状，表內未列出其a/b值。

由实驗結果及其与理論結果的比較，可初步看出以下几点：

1.实驗結果証实了不等曲率双曲扁壳失稳曲面之周界可視为一椭圓，其半軸长之比与其相应曲率开平方成反比。

2.模型实驗得出的临界荷載与理論計算結果大体上是接近的。这說明本文提供的計算理論基本上是能反映实际情况的。模型A_3的临界荷載略为偏高，估計可能是壳体制作时曲率偏高所致。

3.壳面原始缺陷的存在将大大降低壳体的稳定性，如模型A_4之临界荷載只达到模型A_1～A_3临界荷載的51～60%，因此这是壳体設計及施工中必須严加注意的一个問題。

4.由图6.3与图6.4看出，壳面失稳区域一部分已靠近壳体边界，估計这已不是壳体开始失稳瞬間的失稳区，而是随后扩大了的結果。

七、結　論

1.对于建筑中常用尺寸的双曲扁壳，局部失稳之临界荷載远低于总体失稳时的上临荷載，因此在設計中局部失稳是需要考虑的控制因素。

2.椭圓抛物面双曲扁壳局部失稳区域的周界可視为一椭圓。通过壳体失稳临界荷載計算的比較及模型实驗，得出了其半軸长之比可按与相应的曲率开平方成反比来考虑。

3.将壳体失稳区域作为变量处理，提出考虑边界处位移变分不为零时的广义伽辽金变分方程，用此法求得了与能量法一致的結果。由于前者的計算常較簡单，可以推广应用。

4.不等曲率双曲扁壳临界荷載公式的系数不是一个常量，而与曲率比有关。当$\frac{k_x}{k_y}=1\sim2$（$\lambda=0\sim1$），系数由0.288增至0.331时，壳体的实际临界荷載值由$0.288Ek_x^2h^2$降为$0.331Ek_x(0.5k_x)h^2=0.1655Ek_x^2h^2$。这表明当一个方向曲率保持不变，另一方向曲率减小时，壳体的临界荷載将相应地降低。

本文所研究的問題只是短期荷載作用下的非线性弹性稳定性，今后尙須結合材料的特征，加荷龄期以及壳体弹塑性稳定等問題进一步进行研究，以获得更为完善的研究成果。

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